Math 575
Fall 2018
Solutions to Problem Set # 4

(1) (p. 48, 1) (**Not graded**) Suppose that $\{a_n\}$ converges to zero, and that $s_{2n} = \sum_{k=1}^{2n} a_k$ converges to a limit s. We claim that s_n also converges to s. Let $\varepsilon > 0$ be given. We can find $N \in \mathbb{N}$ so that $|a_n| < \varepsilon/2$ for all $n \geq N$, and $|s_{2n} - s| < \varepsilon$ for all $n > N/2$ (i.e., for all even $n > N_1$). If $n \geq N$ is odd we may write $n = 2m+1$ with $m > N/2$ and estimate

$$|s_{2m+1} - s| \leq |s_{2m+1} - s_{2m}| + |s_{2m} - s| < \varepsilon.$$

This shows that $|s_n - s| < \varepsilon$ for all $n \geq N$, so $s_n \to s$.

(2) (pp. 53-54, 19) (**4 points**)
(a) (**2 points**) Apply the integral test to $f(x) = 1/(x(\log x)^a)$. We compute, using the substitution $u = \log x$, $du = dx/x$

$$\begin{aligned}\int_2^R \frac{1}{x(\log x)^a}\,dx &= \int_{\log 2}^{\log R} u^{-a}\,du \\ &= \frac{1}{1-a}\left[(\log R)^{1-a} - (\log 2)^{1-a}\right]\end{aligned}$$

which is bounded as $R \to \infty$ provided $a > 1$. Alternatively, one may use the 2^m test and exploit the fact that $\sum_{m=1}^{\infty} m^{-a}$ converges if and only if $a > 1$.

(b) (**2 points**) Apply the integral test and compute carefully:

$$\begin{aligned}\int_3^R \frac{1}{x(\log x)(\log\log x)^a}\,dx &= \int_{\log 3}^{\log R} \frac{1}{u(\log u)^a}\,du \\ &= \int_{\log\log 3}^{\log\log R} \frac{1}{v^a}\,dv \\ &= \frac{1}{1-a}\left[(\log\log R)^{1-a} - (\log\log 3)^{1-a}\right)\end{aligned}$$

The integral (and hence the sum) only converge if $a > 1$. (Why was the lower limit set to 3 rather than 2 in this problem?)

One can also use the 2^m test iteratively to get to the same conclusion.

(3) (pp. 53-54, 24) (**6 points**) Suppose that $\{a_m\}$ is nonincreasing and has limit zero. We will prove that (a) if $\sum_{m=1}^{\infty} 2^m a_{2^m}$ converges, then $\sum_{k=1}^{\infty} a_k$ converges, and (b) if $\sum_{m=1}^{\infty} 2^m a_{2^m}$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

(a) (**3 points**) Suppose $\sum_{m=1}^{\infty} 2^m a_{2^m}$ converges to a nonnegative real number S, and denote by s_n the nth partial sum $\sum_{k=1}^{n} a_k$. Observe that the sequence of partial sums is monontone nondecreasing, so it suffices to show that $\{s_n\}$ is bounded above. Indeed, by monotonicity, it suffices to show that $\{s_{2^n}\}$ is bounded above. But

$$\begin{aligned} s_{2^n} &= \sum_{\ell=1}^{n} \left(\sum_{k=2^{\ell-1}}^{2^\ell} a_k \right) \\ &\le \sum_{\ell=1}^{n} 2^{\ell-1} a_{2^{\ell-1}} \\ &\le S \end{aligned}$$

(b) (**3 points**) Suppose $\sum_{m=1}^{\infty} 2^m a_{2^m}$ diverges. Since $\{s_k\}$ is a monotone increasing sequence, it suffices to show that $\{s_{2^n}\}$ diverges. But

$$\begin{aligned} s_{2^n} &= \sum_{\ell=1}^{n} \left(\sum_{k=2^{\ell-1}}^{2^\ell} a_k \right) \\ &\ge \sum_{\ell=1}^{n} \left(\sum_{k=2^{\ell-1}}^{2^\ell} a_{2^\ell} \right) \\ &= \frac{1}{2} \sum_{\ell=1}^{n} 2^\ell a_{2^\ell} \end{aligned}$$

so $s_{2^n} \to \infty$ as $n \to \infty$.