Math 641
Final Problem Set
Connections and Curvature
Solutions

1. (a) From the relation $\left\langle \overline{E}_j, \overline{E}_k \right\rangle = \delta_{jk}$ and the fact that the connection ∇ is compatible with the Riemannian metric, we have

$$\begin{aligned} \overline{E}_i \left\langle \overline{E}_j, \overline{E}_k \right\rangle &= \left\langle \nabla_{\overline{E}_i} E_j, E_k \right\rangle + \left\langle \overline{E}_j, \nabla_{\overline{E}_i} \overline{E}_k \right\rangle \\ &= 0 \end{aligned}$$

so that, wrting $\nabla_{\overline{E}_r} \overline{E}_s = \sum_{m=1}^n \overline{\Gamma}^m_{rs} \overline{E}_m$, recalling that $\overline{\Gamma}^m_{rs} = \overline{\Gamma}^m_{sr}$ owing to the symmetry of the connection, and using orthonormality, we recover

$$\overline{\Gamma}^k_{ij} + \overline{\Gamma}^j_{ik} = 0.$$

(b) Let us write $A(x) = \exp(B(x))$ where $x \mapsto B(x)$ is a smooth map taking values in the $n \times n$ antisymmetric matrices with $B(0) = 0$. It follows from the discussion in class that $x \mapsto A(x)$ is a smooth map with values in $SO(n)$ for x in a neighborhood of zero. If X is any vector field defined in a neighborhood of $0 \in \mathbb{R}^n$, we have $X(0)A = X(0)B$ since $A(0) = I$. If $\alpha^j_i(x)$ is the ij component of A, and $\beta^j_i(x)$ is the ij component function of B, it follows that

$$X(0)\alpha^j_i = X(0)\beta^j_i$$

so that by the antisymmetry of B,

$$X(0)\alpha^j_i + X(0)\alpha^j_i = 0$$

as claimed.

(c) Let $\mathbf{x} : U \subset \mathbb{R}^n \to M$ be a coordinate system with $\mathbf{x}(0) = p$. We seek vector fields

$$E_i(q) = \sum_{j=1}^n \gamma^j_i(q) E_j(q)$$

defined for q in a neighborhood of p, where γ^j_i are the components of a matrix in $SO(n)$ with $\gamma^j_i(p) = \delta_{ij}$. Let us set $\alpha^j_i(x) = \gamma^j_i(\mathbf{x}(x))$, and by a slight abuse of notation we'll denote by $\overline{E}_j(0)f$ the number $\left[\left(\overline{E}_j(p)f\right)\circ\mathbf{x}\right](0)$ if f is a smooth function defined near p. We wish to choose α_i^j so that

$$\nabla_{E_i}E_j(p)=0.$$

We compute

$$\begin{aligned}
\nabla_{E_i}E_j(p) &= \nabla_{\sum_r \alpha_i^r \overline{E}_r}\left(\sum_s \alpha_j^s \overline{E}_s\right)\\
&= \sum_{r,s}\alpha_i^r(0)\left(\left[\overline{E}_r(0)\alpha_j^s\right]\overline{E}_s+\alpha_j^s(0)(\nabla_{\overline{E}_r}\overline{E}_s)(p)\right)\\
&= \sum_s \overline{E}_i(0)\alpha_j^s\overline{E}_s(p)+\left(\nabla_{\overline{E}_i}\overline{E}_j\right)(p)
\end{aligned}$$

Taking inner products with $\overline{E}_k(p)$ we get

$$\overline{E}_i(0)\alpha_j^k+\overline{\Gamma}_{ij}^k(p)=0$$

for $1\leq i\leq n$ and each α_j^k. We can prescribe these conditions on the derivatives of the α_j^k provided that they are consistent with the antisymmetry condition $\alpha_j^k+\alpha_k^j=0$. This follows from part (a).

2. We recall the formulas

$$\Gamma_{ij}^k=\frac{1}{2}\sum_m g^{km}\left(g_{im,j}+g_{mj,i}-g_{ij,m}\right) \tag{1}$$

and

$$\begin{aligned}
R_{ijks} &= \sum_m g_{sm}R_{ijk}^m \qquad (2)\\
R_{ijk}^m &= \sum_\ell\left(\Gamma_{ik}^\ell\Gamma_{j\ell}^s-\Gamma_{jk}^\ell\Gamma_{i\ell}^s\right)+\frac{\partial\Gamma_{ik}^s}{\partial x_j}-\frac{\partial\Gamma_{jk}^s}{\partial x_i}
\end{aligned}$$

(a) If $g_{ij}=e^{-2f}\delta_{ij}$ then in any dimension we have

$$g_{ij,k}:=\frac{\partial g_{ij}}{\partial x_k}=-2f_ke^{-2f}\delta_{ij}$$

where

$$f_k = \partial f / \partial x_k.$$

Using the formula (1), we have

$$\begin{aligned}\Gamma_{ij}^{k} &= \frac{1}{2} e^{2f} \left(-2e^{-2f}\right) \left(\delta_{ik} f_j + \delta_{jk} f_i - \delta_{ij} f_k\right) \\ &= \delta_{ij} f_k - \delta_{ik} f_j - \delta_{jk} f_i\end{aligned}$$

Using this formula in two dimensions we can compute

$$-\Gamma_{11}^{1} = \Gamma_{22}^{1} = f_1, \quad \Gamma_{12}^{1} = \Gamma_{22}^{2} = -f_2 \tag{3}$$

and obtain the remaining connection coefficients by symmetry.

(b) First of all, recalling the symmetry relations

$$\begin{aligned}R_{ijks} &= -R_{jiks} \\ R_{ijks} &= -R_{ijsk}\end{aligned}$$

it is clear that R_{ijks} can be nonzero if and only if $i \neq j$ and $k \neq s$. From the same symmetry relations we have $R_{1212} = -R_{2112} = R_{2121} = -R_{1221}$ so the only nonzero component up to sign is R_{1212}. Using the formula (2) we have

$$R_{1212} = e^{-2f} \left\{ T_1 + T_2 \right\} \tag{4}$$

where

$$\begin{aligned}T_1 &= \underbrace{\left(\Gamma_{11}^{1}\Gamma_{21}^{2} - \Gamma_{21}^{1}\Gamma_{11}^{2}\right)}_{\ell=1} + \underbrace{\left(\Gamma_{11}^{2}\Gamma_{22}^{2} - \Gamma_{21}^{2}\Gamma_{12}^{2}\right)}_{\ell=2} \\ T_2 &= \frac{\partial}{\partial x_2}\Gamma_{11}^{2} - \frac{\partial}{\partial x_1}\Gamma_{21}^{2}\end{aligned}$$

Combining these formulas with (3) we get

$$T_1 = 0, \quad T_2 = \Delta f$$

where $\Delta f = \partial^2 f / \partial x_1^2 + \partial^2 f / \partial x_2^2$. It now follows from (4) that

$$R_{1212} = e^{-2f} \Delta f \tag{5}$$

as claimed.

(c) The metric on the upper half-plane $\mathbb{H}^2$ is given by

$$g_{ij} = x_2^{-2}\delta_{ij}.$$

For future use we note the following general formula for metrics of the form $g_{ij} = e^{-2f}\delta_{ij}$ in n dimensions. If $X_i = \partial/\partial x_i$ for $i = 1, \cdots, n$ the vector fields $Y_i = e^f X_i$ are orthonormal. Thus $|Y_i \wedge Y_j| = 1$ if $i \neq j$. The plane spanned by Y_i and Y_j then has sectional curvature

$$\begin{aligned} K_{ijij} &= \frac{R(Y_i, Y_j.Y_i, Y_j)}{|Y_i \wedge Y_j|^2} \\ &= R(e^f X_1, e^f X_2, e^f X_1, e^f X_2) \\ &= e^{4f} R_{ijij}. \end{aligned}$$

It now follows from (5) that if $n = 2$ and $g_{ij} = e^{-2f}\delta_{ij}$, the formula

$$K_{1212} = e^{2f}\Delta f \tag{6}$$

holds. In our case $f = \log x_2$, $e^{2f} = x_2^2$ and

$$\Delta f = \frac{\partial^2 f}{\partial x_2^2} = -\frac{1}{x_2^2}$$

so that

$$K_{1212} = -1$$

as claimed.

3. This problem concerns the other model space with nonzero curvature, the sphere S^2.

(a) To construct the conformal map note that the line through $(0, 0, -1)$ and $(x_1, x_2, 0)$ has the parametric form

$$(y_1, y_2, y_3) = (tx_1, tx_2, -1 + t)$$

and the image point (y_1, y_2, y_3) on the unit sphere is determined by the condition that $|y|^2 = 1$ or

$$t^2 |x|^2 + (1 - t)^2 = 1.$$

Solving this quadratic equation for t yields the roots $t = 0$ (corresponding to $(0, 0, -1)$) and $t = 2/\left(1 + |x|^2\right)$. Picking the latter root we get

$$\psi(x) = \left(\frac{2x_1}{1 + |x|^2}, \frac{2x_2}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2}\right).$$

Note that $\psi(0) = (0, 0, 1)$ and $\lim_{|x|\to\infty} \psi(x) = (0, 0, -1)$ as expected.

(b) To compute the Jacobian matrix of ψ we denote by (y_1, y_2, y_3) the component functions of ψ and compute the partials

$$\begin{aligned}
\frac{\partial y_1}{\partial x_1} &= \frac{2}{1 + |x|^2} - \frac{4x_1^2}{\left(1 + |x|^2\right)^2} \\
\frac{\partial y_1}{\partial x_2} &= -\frac{4x_1 x_2}{\left(1 + |x|^2\right)^2} = \frac{\partial y_2}{\partial x_1} \\
\frac{\partial y_2}{\partial x_2} &= \frac{2}{1 + |x|^2} - \frac{4x_2^2}{\left(1 + |x|^2\right)^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial y_3}{\partial x_1} &= \frac{-4x_1}{\left(1 + |x|^2\right)^2} \\
\frac{\partial y_3}{\partial x_2} &= \frac{-4x_2}{\left(1 + |x|^2\right)^2}
\end{aligned}$$

which give the claimed form for a and b^T. It is not difficult to compute that

$$a^T a = \frac{4}{\left(1 + |x|^2\right)^4} \begin{pmatrix} (1+t)^2 + 4s^2 & -4s \\ -4s & (1-t)^2 + 4s^2 \end{pmatrix}$$

where $t = x_2^2 - x_1^2$ and $s = x_1 x_2$, while

$$b b^T = \frac{16}{\left(1 + |x|^2\right)^4} \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix}.$$

It now follows that

$$\begin{aligned} J^T J &= a^T a + b b^T \\ &= \frac{4}{\left(1+|x|^2\right)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

as claimed. To see that the matrices g_{ij} and $J^T J$ coincide, note that if $\{e_1, e_2\}$ are the standard basis on $\mathbb{R}^2$, then

$$\begin{aligned} g_{ij} &= \langle J e_i, J e_j \rangle_{\mathbb{R}^3} \\ &= (J e_i)^T (J e_j) \\ &= e_i^T J^T J e_j \\ &= \left(J^T J\right)_{ij} . \end{aligned}$$

(c) Thanks to formula (6) this is an easy computation. We have $e^{-2f} = 4/(1+|x|^2)$ so

$$f = \frac{1}{2} \log \left(\frac{1+|x|^2}{4} \right)$$

and

$$\begin{aligned} \Delta f &= \nabla \cdot (\nabla f) \\ &= \nabla \cdot \left(\frac{2x}{1+|x^2|} \right) \\ &= \frac{4}{1+|x|^2} + 2x \cdot \frac{-2x}{\left(1+|x|^2\right)^2} \\ &= \frac{4}{\left(1+|x|^2\right)^2} . \end{aligned}$$

Hence

$$\begin{aligned} K_{1212} &= e^{2f} \Delta f \\ &= \left(\frac{1+|x|^2}{2} \right)^2 \left(\frac{4}{\left(1+|x|^2\right)^2} \right) \\ &= 1 \end{aligned}$$

as claimed.