Math 641
Final Problem Set
Due Monday, December 15

1. The purpose of this problem is to show the existence of local geodesic frames. Suppose that M is a smooth manifold with a Riemannian metric $\langle \cdot , \cdot \rangle$ and a compatible connection ∇.

   (a) Show that if $\{\overline{E_i}\}_{i=1}^n$ is an orthonormal set of vector fields, then the connection coefficients Γ_{ij}^k defined by
   $$\nabla_{\overline{E}_i}\overline{E}_j = \sum_{k=1}^n \Gamma_{ij}^k \overline{E}_k$$
   satisfy the condition
   $$\Gamma_{ij}^k + \Gamma_{ik}^j = 0.$$
   Hint: Use the fact that the connection is compatible with the Riemannian metric.

   (b) Suppose that $x \mapsto A(x)$ is a smooth mapping from a neighborhood U of $\mathbb{R}^n$ containing 0 into the $SO(n)$ matrices, defined by the relations
   $$A^t A = I$$
   $$\det(A) = 1.$$
   Let $\alpha_i^j(x)$ be the component functions of $A(x)$ and suppose that $\alpha_i^j(0) = \delta_i^j$. (so that $A(0)$ is the identity matrix). Show that for any vector field X on U,
   $$X(0)\left(\alpha_i^j\right) + X(0)\left(\alpha_j^i\right) = 0.$$

   (c) Let $p \in M$. Show that there is a neighborhood U of p and a set of orthonormal vector fields $\{E_i\}_{i=1}^n$ with the property that
   $$\nabla_{E_i} E_j(p) = 0. \qquad (1)$$
   Hint: There is an orthonormal frame $\{\overline{E_i}\}_{i=1}^n$ defined in a neighborhood U of p. Obtain the desired frame by taking
   $$E_i = \sum \alpha_i^j \overline{E_j}$$

where the α_i^j are smooth functions so chosen that the α_i^j are the components of a matrix in $SO(n)$. Find a condition on the first derivatives of α_i^j at p so that (1) holds, and verify that this condition is compatible with the constraint that α_i^j are the components of a matrix in $SO(n)$. Parts (a) and (b) are relevant.

2. Suppose that M is a surface so that, in local coordinates (x_1, x_2):

$$g_{11} = g_{22} = e^{-2f}$$
$$g_{12} = g_{21} = 0$$

where f is a smooth function of x_1 and x_2.

(a) Find the connection coefficients for the coordinate vector fields $\partial/\partial x_1$ and $\partial/\partial x_2$. *Hint*: By symmetry of the connection $\Gamma_{12}^k = \Gamma_{21}^k$ for $k = 1, 2$ and by problem 1(a) we have $\Gamma_{22}^1 = -\Gamma_{12}^2$ and $\Gamma_{12}^1 = -\Gamma_{11}^2$. So, really, it suffices to compute Γ_{11}^1, Γ_{12}^1, Γ_{22}^1, Γ_{22}^2 - down from 8 connections coefficients to four!

(b) Show that the only nonzero component of the curvature tensor R_{ijks} is R_{1212} (up to the usual symmetries) and show that

$$R_{1212} = e^{-2f}\Delta f$$

where

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}.$$

(c) Use this result to show that the hyperbolic metric on the upper half-plane $H = \{(x_1, x_2) : x_1 \in \mathbb{R},\ x_2 > 0\}$ with metric

$$g_{11} = g_{22} = x_2^{-2},\ g_{12} = 0$$

has constant curvature -1.

3. This problem concerns the two-dimensional sphere, regarded as an embedded submanifold of $\mathbb{R}^3$ given by the solution set of the equation $x_1^2 + x_2^2 + x_3^2 = 1$.. For ten points extra credit, carry out this problem for the n-dimensional sphere regarded as an embedded submanifold of $\mathbb{R}^{n+1}$.

(a) Consider the map $\psi : \mathbb{R}^2 \to S^2$ by stereographic projection: if $x \in \mathbb{R}^2$, $\psi(x)$ is the point on the sphere which intersects a line through the point $(-1, 0, 0)$ (south pole) and $(x_1, x_2, 0)$. Show that

$$\psi(x) = \left(\frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right).$$

(b) Let A denote the Jacobian matrix of the map ψ. Show that the metric induced on $\mathbb{R}^2$ by the Euclidean metric on the sphere is given by $A^T A$, and compute that

$$A = \begin{bmatrix} a \\ b^T \end{bmatrix}$$

where a is the 2×2 matrix

$$\frac{\partial \psi_i}{\partial x_j} = \delta_{ij} \frac{2}{1 + |x|^2} - \frac{4 x_i x_j}{(1 + |x|^2)^2}$$

for $1 \le i, j \le 2$, and b^T is the row vector

$$\left(\frac{-4x_1}{\left(1 + |x|^2\right)^2}, \frac{-4x_2}{\left(1 + |x|^2\right)^2} \right)$$

Conclude that

$$A^T A = a^T a + b b^T$$

and compute $A^T A$. You should get

$$A^T A = \delta_{ij} \frac{4}{\left(1 + |x|^2\right)^2}$$

(c) The metric on the sphere in these coordinates is a conformal metric of the type considered in problem 2 with

$$e^{-2f} = \frac{4}{\left(1 + |x|^2\right)^2}.$$

Using the results of problem 2, show that the sphere has curvature

$$K_{1212} = +1.$$