1 Introduction

These notes discuss geodesics on a Riemannian manifold from the point of view of the Calculus of Variations. Morally, geodesics should be paths $\gamma : [a, b] \to M$ which minimize the length functional

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\| \; dt \tag{1}$$

where $\|v\|$ is the norm of a vector $v \in T_pM$. A related functional is the *energy functional*

$$E(\gamma) = \frac{1}{2}\int_a^b \|\gamma'(t)\|^2 \; dt. \tag{2}$$

Both of these equations define a *functional*, i.e., a mapping from the "space of paths" to the nonnegative real numbers. Just as the gradient of a smooth function of several variables vanishes at local extrema, , so the *first variation* (defined below) of a functional vanishes at local extrema. Note however that in both cases, the vanishing of the first variation is necessary but not sufficient (think of saddle points for a function like $f(x, y) = x^2 - y^2$).

A full development of the Calculus of Variations is beyond the scope of this course. We would like however to show the relationship between geodesics as curves defined by the condition

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$$

and minima of the length functional. To begin with we need to introduce some ideas of the Calculus of variations.

2 A Short Course in the Calculus of Variations, Part I

We wish to study how a functional on paths such as $\ell(\gamma)$ or $E(\gamma)$ changes if the path is varied; in applications we're interested in all paths connecting two points, so that the endpoints are fixed. To define a path with fixed endpoints consider the class $\mathcal{C}$ of smooth mappings $f : [a, b] \to \mathbb{R}^n$ with $f(a) = \mathbf{x}_1$ and $f(b) = \mathbf{x}_2$ fixed. We can think of the f's as parameterizing smooth curves from $\mathbf{x}_1$ to $\mathbf{x}_2$. A variation is a smooth mapping

$$\begin{aligned} \alpha : (-\delta, \delta) \times [a, b] &\to \mathbb{R}^n \\ (u, t) &\mapsto \alpha(u, t) \end{aligned}$$

and we require that

$$\alpha(u, a) = \mathbf{x}_1 \tag{3}$$
$$\alpha(u, b) = \mathbf{x}_2 \tag{4}$$

so that end points are fixed. We'll denote by α_i the ith component function of α. Thus, for each fixed u, the function

$$\alpha_u(t) = \alpha(u,t)$$

is a path in $\mathcal{C}$. For later use, we note that

$$\frac{\partial \alpha}{\partial u}(u,a) = \frac{\partial \alpha}{\partial u}(u,b) = 0$$

owing to (3)-(4).

Suppose we are given a functional of the form

$$J(f) = \int_a^b F(t, f(t), f'(t))\ dt \tag{5}$$

where

$$F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$
$$(t,x,y) \to F(t,x,y)$$

We wish to find a condition on f (in the form of a differential equation) that minimizes (or maximizes) the functional J. over the class $\mathcal{C}$. Two important examples of such functionals on paths are the length functional

$$\ell(f) = \int_a^b \|f'(t)\|\ dt$$

and the energy functional

$$E(f) = \frac{1}{2}\int \|f'(t)\|^2\, dt$$

which, as we shall see, are closely related.

The map $J : \mathcal{C} \to \mathbb{R}$ is presumed to be stationary at any maxima or minima. A function f is a stationary point of J if for any variation α with $\alpha(0,t) = f(t)$, we have

$$\left.\frac{d}{du}\right|_{u=0} J(\alpha_u) = 0 \tag{6}$$

where

$$\alpha_u(t) = \alpha(u,t)$$

is a path from $\mathbf{x}_1$ to $\mathbf{x}_2$. . We can use the stationarity condition to derive a system of ordinary differential equations, called the *Euler-Lagrange Equations*, for the variational problem. They are analogous to the statement that the derivative of a function at a local maximum or minimum is zero; a solution of the Euler-Lagrange equations may be a local maximum, a local minimum, or neither, and there is a "second derivative test" for local maxima and minima of J to which we will return later.

To derive the Euler-Lagrange equations, let us compute

$$\frac{d}{du}F(t, \alpha_u(t), \alpha_u'(t))$$

where

$$\alpha_u'(t) = \frac{\partial \alpha}{\partial t}(u, t).$$

From the chain rule we have

$$\frac{d}{du}F(t, \alpha_u(t), \alpha_u'(t)) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t, \alpha_u(t), \alpha_u'(t)) \frac{\partial \alpha_i}{\partial u}(u, t) + \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(t, \alpha_u(t), \alpha_u'(t)) \frac{\partial^2 \alpha_i}{\partial u \, \partial t}(u, t)$$

so that differentiating under the integral sign in (6) we have

$$\frac{d}{du}J(\alpha_u) = \int_a^b \left\{ \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t, \alpha_u(t), \alpha_u'(t)) \frac{\partial \alpha_i}{\partial u}(u, t) \right\} dt + \int_a^b \left\{ \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(t, \alpha_u(t), \alpha_u'(t)) \frac{\partial^2 \alpha_i}{\partial u \, \partial t}(u, t) \right\} dt. \tag{7}$$

Using the fact that

$$\frac{d}{dt}\left\{ \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(t, \alpha_u(t), \alpha_u'(t)) \frac{\partial \alpha_i}{\partial u}(u, t) \right\} = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial F}{\partial y_i}(t, \alpha_u(t), \alpha_u'(t)) \right) \frac{\partial \alpha_i}{\partial u}(u, t) + \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(t, \alpha_u(t), \alpha_u'(t)) \frac{\partial^2 \alpha_i}{\partial u \, \partial t}(u, t)$$

and exploiting the fact that

$$\frac{\partial \alpha}{\partial u}(u, a) = \frac{\partial \alpha}{\partial u}(u, b) = 0$$

in virtue of (3) and (4), we can recast (7) as

$$\frac{d}{du}J(\alpha_u) = \int_a^b \sum_{i=1}^{n} \left\{ \frac{\partial F}{\partial x_i}(t, \alpha_u(t), \alpha_u'(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial y_i}(t, \alpha_u(t), \alpha_u'(t)) \right) \right\} \frac{\partial \alpha_i}{\partial u}(u, t) \, dt$$

and setting $u = 0$ we get

$$0 = \int_a^b \sum_{i=1}^{n} \left\{ \frac{\partial F}{\partial x_i}(t, f(t), f'(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial y_i}(t, f(t), f'(t)) \right) \right\} \frac{\partial \alpha_i}{\partial u}(0, t) \, dt$$

for any stationary point f. Since the variation is arbitrary, we conclude that any stationary point obeys the equations

$$\frac{\partial F}{\partial x_i}(t, f(t), f'(t)) - \frac{d}{dt}\left(\frac{\partial F}{\partial y_i}(t, f(t), f'(t)) \right) = 0. \tag{8}$$

These equations are the *Euler-Lagrange equations* for the variational problem.

3 The Energy Functional and the Length Functional

It would be natural to compute the Euler-Lagrange equations for the length functional on paths in a Riemannian manifold, but we'll compute the Euler-Lagrange equations for the energy functional first because, as we'll see, these are very nice equations and it will turn out that every stationary point of the energy functional is also a stationary point of the length functional.

Let's consider a local coordinate chart $\mathbf{x} : U \subset \mathbb{R}^n \to M$ and the usual coordinate vector fields

$$\left.\frac{\partial}{\partial x_i}\right|_{\mathbf{x}(x)} = d\mathbf{x}_x(e_i).$$

Let us set

$$g_{ij}(x) = \left\langle \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{x}(x)}, \left.\frac{\partial}{\partial x_j}\right|_{\mathbf{x}(x)} \right\rangle_{\mathbf{x}(x)}$$

and write $Y \in T_{\mathbf{x}(x)}M$ as

$$Y = \sum_{j=1}^{n} y^j \left.\frac{\partial}{\partial x_j}\right|_{\mathbf{x}(x)}.$$

In this case,

$$F(t,x,y) = \sum_{j,k=1}^{n} g_{jk}(x) y_j y_k$$

(thus F is actually independent of t) so that

$$\frac{\partial F}{\partial x_i}(t,x,y) = \sum_{j=1}^{n} \frac{\partial g_{jk}}{\partial x_i} y_j y_k$$

$$\frac{\partial F}{\partial y_i}(t,x,y) = 2\sum_{k=1}^{n} g_{ik} y_k$$

and if

$$f(t) = (x_1(t), \cdots, x_n(t))$$

then

$$\frac{d}{dt}\left(\frac{\partial F}{\partial y_i}(t, f(t), f'(t))\right) = 2\sum_{k,\ell=1}^{n} \frac{\partial g_{ik}}{\partial x_\ell}(x(t)) \frac{dx_\ell}{dt}\frac{dx_k}{dt} + 2\sum_{k=1}^{n} g_{ik}(x(t)) \frac{d^2 x_k}{dt^2}.$$

Let's write g_{ik} for $g_{ik}(x(t))$, $\dot{x}_i$ for dx_i/dt, and $\ddot{x}_i$ for d^2x_i/dt^2. Putting it all together we get the equations

$$\sum_k \frac{\partial g_{jk}}{\partial x_i} \dot{x}_j \dot{x}_k = 2\sum_{k,\ell} \frac{\partial g_{ik}}{\partial x_\ell} \dot{x}_\ell \dot{x}_k + 2\sum_k g_{ik} \ddot{x}_k$$

or (renaming some indices and rearranging)

$$\sum_{\ell} g_{m\ell}\ddot{x}_\ell = -\sum_{j,k} \frac{\partial g_{mj}}{\partial x_\ell}\dot{x}_j\dot{x}_k + \frac{1}{2}\sum_{j,k}\frac{\partial g_{jk}}{\partial x_m}\dot{x}_j\dot{x}_k$$

We wish to solve for $\ddot{x}_k$. Let g^{jk} be the matrix inverse of g_{jk}. If we multiply both sides of this equation by g^{im} and sum over m, we get

$$\ddot{x}_i = -\sum_{m,j,k} g^{im}\frac{\partial g_{mj}}{\partial x_\ell}\dot{x}_j\dot{x}_k + \frac{1}{2}\sum_{m,j,k} g^{im}\frac{\partial g_{jk}}{\partial x_m}\dot{x}_j\dot{x}_k$$

and we can exploit the symmetry $g_{mj} = g_{jm}$ to rewrite this as

$$\ddot{x}_i = -\frac{1}{2}\sum_m g^{im}\left(\sum_{,j,k}\left\{\frac{\partial g_{mj}}{\partial x_\ell} + \frac{\partial g_{jm}}{\partial x_\ell} - \frac{\partial g_{jk}}{\partial x_m}\right\}\dot{x}_j\dot{x}_k\right)$$

which we recognize as the geodesic equations.

Hence, any stationary path for the energy functional is a geodesic, and hence has constant speed. This means that, for a stationary path,

$$\frac{d}{dt}\left(F(t, f(t), f'(t)\right) = 0$$

since $F(t)$ is the (squared) speed of the geodesic at time t.

What about the length functional? The length functional takes the form

$$\ell(\gamma) = \int_a^b F_1(t, f, f')\ dt$$

where

$$F_1(t, x, y) = F(t, x, y)^{1/2}.$$

Using this fact, one can show that the Euler-Lagrange equations for the length functional are equivalent to

$$\frac{\partial F}{\partial x_i}(t, f(t), f'(t)) = \frac{d}{dt}\left(\frac{\partial F}{\partial y_i}(t, f(t), f'(t))\right) - \frac{1}{2}F^{-1}\dot{F} \tag{9}$$

If f solves the Euler-Lagrange equations for the energy functional, then $\dot{F} = 0$ and

$$\frac{\partial F}{\partial x_i}(t, f(t), f'(t)) = \frac{d}{dt}\left(\frac{\partial F}{\partial y_i}(t, f(t), f'(t))\right).$$

Thus:

Theorem 1 *Suppose that $x(t)$ is a stationary point of the energy functional. Then $x(t)$ is also a stationary point of the length functional.*