Math 641
Homework 1
Due Friday, September 5

These problems all concern basic notions of differential calculus. A good reference is the book by Michael Spivak, *Calculus on Manifolds*, chapter 2 .See particularly the discussion of the chain rule for a composition of functions, which we will review on Wednesday September 3.

1. Recall that a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x = a$, if there is a linear mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ with the property that

$$\lim_{\|h\| \to 0} \frac{\|f(a+h) - f(a) - T(h)\|_{\mathbb{R}^m}}{\|h\|_{\mathbb{R}^n}} = 0.$$

Denote by f^i the component functions of f, $1 \leq i \leq m$. That is

$$f(x) = \big(f^1(x), f^2(x), \cdots, f^m(x)\big)$$

for each $x \in U$. Show that all of the partial derivatives

$$D_j f^i(a) = \lim_{h \to 0} \frac{f^i(a + he_j) - f^i(a)}{h}$$

exist, where e_j is the jth basis vector of the usual basis for $\mathbb{R}^n$. How are they computed in terms of the matrix of T with respect to the standard basis? Prove that your formula is correct.

2. Suppose that $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ is a differentiable curve, and $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a C^1 function defined on an open set U containing $\gamma(I)$. Using the chain rule, show that for any $t \in I$,

$$\frac{d}{dt}(f \circ \gamma)(t) = (\nabla f)(\gamma(t)) \cdot \gamma'(t)$$

where $\nabla f(a)$ is the gradient of f at a and $\gamma'(t)$ is the tangent vector to the curve γ at $\gamma(t)$.

3. The $(n-1)$-dimensional sphere can be realized as the zero set of the function
$$F(x_1, \cdots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - 1$$
Suppose that $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ is a curve that obeys the equation
$$F(\gamma(t)) = 0$$
Show that $\gamma(t) \cdot \gamma'(t) = 0$. Interpret this result geometrically.