Math 641
Homework 1
Solutions

In the solutions we'll use the notation $\langle x, y\rangle$ to denote the dot product of Euclidean vectors x and y.

1. First, observe that if $\pi_i : \mathbb{R}^m \to \mathbb{R}$ is the projection onto the ith coordinate and
$$g(h) = f(a+h) - f(a) - T(h)$$
then
$$(\pi_i \circ g)(h) = f^i(a+h) - f^i(a) - (\pi_i \circ T)(h).$$
Next observe that, as $|(\pi_i \circ g)(h)| \leq \|g(h)\|_{\mathbb{R}^m}$ it follows that
$$\lim_{\|h\|\to 0} \frac{|f^i(a+h) - f^i(h) - (\pi_i \circ T)(h)|}{\|h\|_{\mathbb{R}^n}} = 0$$
for each i. In particular, taking $h = ke_j$ for a real number k and noting that $T(ke_j) = kT(e_j)$, we have
$$\lim_{k\to 0} \left| \frac{f^i(a+ke_j) - f^i(a)}{k} - (\pi_i \circ T)(e_j)\right| = 0$$
which shows that
$$\lim_{k\to 0} \frac{f^i(a+ke_j) - f^i(a)}{k} = (\pi_i \circ T)(e_j).$$
We observe that the right-hand side is simply the ijth entry of the matrix of T with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. From this we conclude that if f is differentiable at a then all partial derivatives exist that $D_j f^i(a)$ is given by the ijth entry of the Jacobian matrix.

2. This is a straightforward application of the chain rule since
$$f'(a)(h) = \langle (\nabla f)(a), h\rangle .$$
Thus
$$\begin{aligned}\frac{d}{dt} f(\gamma(t)) &= f'(\gamma(t)) \cdot \gamma'(t) \\ &= \langle \nabla f(\gamma(t), \gamma'(t)\rangle\end{aligned}$$
Suppose that $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ is a differentiable curve, and $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is a C^1 function defined on an open set U containing $\gamma(I)$.

3. Since γ lies on the unit sphere we have

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

Differentiating both sides of this identity we get by the "product rule"[1]

$$2 \langle \gamma(t), \gamma'(t) \rangle = 0$$

or equivalently

$$\langle \gamma(t), \gamma'(t) \rangle = 0.$$

This equation has the following geometric interpretation. For each t, the vector $\gamma(t)$ is in the direction normal to the surface of the sphere. Thus, the vector $\gamma'(t)$, which is perpendicular to the normal, is therefore tangent to the surface of the sphere.

[1] You can prove the product rule for $f : I \subset \mathbb{R} \to \mathbb{R}^n$ and $g : I \subset \mathbb{R} \to \mathbb{R}^n$ from the product rule for functions of one variable as follows:

$$\begin{aligned} \frac{d}{dt} \langle f(t), g(t) \rangle &= \frac{d}{dt} \left(\sum_{i=1}^{n} f^i(t) g^i(t) \right) \\ &= \sum_{i=1}^{n} \frac{d}{dt}(f_i(t)) g_i(t) + f_i(t) \frac{d}{dt}(g_i(t)) \\ &= \langle f'(t), g(t) \rangle + \langle f(t), g'(t) \rangle \end{aligned}$$