Math 641
Homework 2
Due Friday, September 12

1. Let S^n denote the zero set of the function

$$F(x) = |x|^2 - 1$$

where $F : \mathbb{R}^{n+1} \to \mathbb{R}$. Use the implicit function theorem to show that there is a neighborhood $U \subset \mathbb{R}^n$ containing 0 and a differentiable map $g : U \to \mathbb{R}$ with $g(0) = 1$ and $F(x, g(x)) = 0$. Also, compute the map g directly. Note that if we define $f(x) = (x, g(x))$ for x in an open neighborhood U of 0 in $\mathbb{R}^n$, we obtain a *local parameterization* of the sphere S^n near its north pole $(0, \ldots, 0, 1)$.

2. Given the set of local parameterizations $f_\alpha : U_\alpha \to S^2$ for the two-dimensional sphere discussed in class, prove the following.

   (a) $f_\alpha : U_\alpha \to S^2$ is a local homeomorphism,
   (b) df_α has maximal rank,
   (c) $\cup_\alpha f_\alpha(U_\alpha) = S^2$.
   (d) $f_\alpha^{-1} \circ f_\beta$ is smooth (give a formula!)

3. (From M. Spivak, *Calculus on Manifolds*, ch. 5.1): Another equivalent definition of a k-dimensional submanifold of $\mathbb{R}^n$ is the following: a subset M of $\mathbb{R}^n$ is a k-dimensional submanifold of $\mathbb{R}^n$ if for every $p \in M$ there is an open set U containing x, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism (i.e., a smoothly invertible smooth map) $h : U \to V$ so that

$$\begin{aligned} h(U \cap M) &= V \cap \left\{\mathbb{R}^k \times \{0\}\right\} \\ &= \{y \in V : y_{k+1} = \cdots = y_n = 0\}. \end{aligned}$$

The *graph* of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x)\}.$$

Using the definition above, prove that the graph of f is an n-dimensional manifold of $\mathbb{R}^{n+m}$ if and only if f is differentiable.