Math 641
Homework 2
Solutions

1. Let S^n denote the zero set of the function

$$F(x) = |x|^2 - 1$$

where $F : \mathbb{R}^{n+1} \to \mathbb{R}$. The north pole of the sphere is the point $(0,1) \in \mathbb{R}^n \times \mathbb{R}$ (we'll use this ordered pair notation for points in $\mathbb{R}^{n+1}$ where the first entry is a point in $\mathbb{R}^n$). It is easy to compute that

$$F'(0,1) = (0 \quad \cdots \quad 0 \quad 1)$$

so that $F'(0,1)$ has rank one. By the implicit function theorem there is a neighborhood U of 0 and a smooth map $g : U \to \mathbb{R}$ so that $g(0) = 1$ and $F(x, g(x)) = 0$. Hence the set

$$\{(x, g(x)) : x \in U\}$$

is contained in S^n. We can compute the map explicitly by solving $|x|^2 - 1 = 0$. Writing $x \in \mathbb{R}^{n+1}$ as (x', x_{n+1}) for $x' \in \mathbb{R}^n$ we get

$$x_{n+1} = \sqrt{1 - |x'|^2}$$

so

$$g(x') = \sqrt{1 - |x'|^2}$$

which is smooth in the region $U = \{x' \in \mathbb{R}^n : |x'| < 1\}$.

2. We can label the local parameterizations as $f_{i,\pm}$ for $i = 1, 2, 3$ where $f_{i,\pm}$ is a function of (u, v) in the open unit disc U and the ith component is $\pm\sqrt{1 - u^2 - v^2}$. Note that the ith coordinate of $f_{i,\pm}$ takes values in $[-1, 0)$ for $f_{i,+}$ and $(0, 1]$ (for $f_{i,-}$).

   (a) Since $f_{i,\pm}$ are smooth injective maps it suffices to show that their inverses are continuous. To do this it suffices to show that the image of an open disc V in U is an open subset of S^2. Since $f_{i,\pm}(U)$ is the intersection of an open cylinder $V \times (-2, 2)$ with S^2, it is a relatively open set in S^2. Alternatively, one can note that the inverse functions are the restrictions to S^2 of projection maps: for example $f_{3,+}^{-1}(x, y, z) = (x, y)$ which is a continuous map from $\mathbb{R}^3$ to $\mathbb{R}^2$. The restriction to S^2 is therefore continuous.

(b) We can compute, for example

$$df_{3,+}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\dfrac{u}{\sqrt{1-u^2-v^2}} & -\dfrac{v}{\sqrt{1-u^2-v^2}} \end{pmatrix}$$

so the right-hand side has maximal rank. More generally the matrix $df_{i,\pm}$ is row-equivalent to a 3×2 matrix with zeros in the ith row and $(1\ \ 0)$ and $(0\ \ 1)$ in the complementary rows. Hence $df_{i,\pm}$ has maximal rank.

(c) Suppose that $p = (x_1, x_2, x_3) \in S^2$. At least one coordinate is nonzero, say x_i, and either $x_i > 0$ or $x_1 < 0$. In the first case $p \in f_{i,-}(U)$ and in the second case, $p \in f_{i,+}(U)$

(d) For $f_{2,+}^{-1} \circ f_{3,+}$ for example we have

$$\left(f_{2,+}^{-1} \circ f_{3,+}\right)(u,v) = \left(u, \sqrt{1-u^2-v^2}\right)$$

which is smooth on U. The remaining compositions are of the same form up to $\pm$ on the radical, permutations of u and v, and of the ordering of entries

3. We'll denote by $\Gamma(f)$ the graph of f.
First, suppose that f is differentiable. The mapping $h(x,y) = (x, y - f(x))$ is a smooth map since f is differentiable and it has smooth inverse $h^{-1}(x,y) = (x, y + f(x))$. On the graph of f we have $h(x,y) = (x,0)$ so this shows that the graph of f is a smooth submanifold of $\mathbb{R}^{n+m}$ in Spivak's sense.
Next, suppose that the graph of f is a smooth submanifold of $\mathbb{R}^{n+m}$ and let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projection onto the second factor. For each $p = (a, f(a))$ belonging to the graph, there is a diffeomorphism h defined in a neighborhood of p so that $(\pi \circ h)(x,y) = 0$ for $(x,y) \in \Gamma(f)$. Since h is a diffeomorphism the map $\pi \circ h$ has a differentiable at (a,b) of maximal rank, so we can apply the Implicit Function Theorem to conclude that there is a differentiable function $g : U \to \mathbb{R}^m$ defined in a neighborhood of a with $g(a) = b$ and $(\pi \circ h)(x, g(x)) = 0$. By the uniqueness assertion of the Implicit Function Theorem and the fact that $\Gamma(f)$ has the same property, we conclude that $f(x) = g(x)$ so that

f is differentiable.

Note: Some students assumed that the map h in the second part takes the form $h(x, y) = (x, y - f(x))$ but this need not be the case. Given a map, not necessarily of that form, which satisfies the condition that $\pi \circ h = 0$ on $\Gamma(f)$, I don't see a way around using the Implicit Function Theorem to construct the function g.