Math 641
Homework 3
Solutions

1. Pick $p \in M$ and let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ with $\mathbf{x}(0) = p$ and $d\mathbf{x}_0$ has maximal rank. The Jacobian matrix takes the form

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

and one of the 2×2 submatrices is nonsingular at $(u, v) = (0, 0)$. If $\partial(x, y)/\partial(u, v)(0, 0) \neq 0$ then by the inverse function theorem we can invert the map

$$(u, v) \mapsto (x(u, v), y(u, v))$$

and obtain the smooth map

$$(x, y) \to (u(x, y), v(x, y))$$

to obtain the local representation

$$(x, y, z(u(x, y), v(x, y))$$

or setting $f(x, y) = z(u(x, y), v(x, y))$ we get a representation of the claimed form. Similarly if $\partial(x, z)/\partial(u, v) \neq 0$ we can solve for the parameters u and v in terms of (x, z) and obtain a local representation of the form $(x, g(x, z), z)$, while if $\partial(y, z)/\partial(u, v) \neq 0$ we obtain a local representation of the from $(h(y, z), y, z)$.

2. This is an easy exercise in topology.

   (a) If $A = M$ then $\mathbf{x}_\alpha^{-1} (\mathbf{x}_\alpha(U_\alpha) \cap M) = U_\alpha$ which is open in $\mathbb{R}^n$, so M is open. If $A = \emptyset$ then $\mathbf{x}_\alpha^{-1}(\mathbf{x}_\alpha(U_\alpha) \cap \emptyset) = \emptyset$ for all α so $\emptyset$ is open.

(b) If $\{A_i\}_{i\in I}$ is a collection of open sets then

$$\mathbf{x}_\alpha^{-1}(\cup_{i\in I}A_i \cap \mathbf{x}_\alpha(U_\alpha)) = \cup_{i\in I}\mathbf{x}_\alpha^{-1}(A_i \cap \mathbf{x}_\alpha(U_\alpha))$$

and since each $\mathbf{x}_\alpha^{-1}(A_i\cap\mathbf{x}_\alpha(U_\alpha))$ is an open subset of $\mathbb{R}^n$, the union gives an open subset of $\mathbb{R}^n$ as well. Hence $\cup_{i\in I}A_i$ is open.

(c) If $\{A_i\}_{i=1}^m$ is a finite collection of open sets then

$$\mathbf{x}_\alpha^{-1}(\cap_{i=1}^n A_i \cap \mathbf{x}_\alpha(U_\alpha)) = \cap_{i=1}^n \mathbf{x}_\alpha^{-1}(A_i \cap \mathbf{x}_\alpha(U_\alpha))$$

and since each $\mathbf{x}_\alpha^{-1}(A_i\cap\mathbf{x}_\alpha(U_\alpha))$ is an open subset of $\mathbb{R}^n$, the union gives an open subset of $\mathbb{R}^n$ as well. Hence $\cap_{i=1}^n A_i$ is open.

3. This problem will test your capacity to stand multiple Greek subscripts. Suppose that M and N are smooth manifolds of respective dimensions m and n with respective atlases $\{(U_\alpha, \mathbf{x}_\alpha)\}$ and $\{(V_\beta, \mathbf{y}_\beta)\}$ and define charts on $M \times N$ by

$$\mathbf{z}_{\alpha\beta}(x, y) = (\mathbf{x}_\alpha(x), \mathbf{y}_\beta(y)).$$

We claim that $\{(U_\alpha \times V_\beta), \mathbf{z}_{\alpha\beta})\}$ is an atlas for $M \times N$. First, note that

$$\begin{aligned}\cup_\alpha \cup_\beta \mathbf{z}_{\alpha\beta}(U_\alpha \times V_\beta) &= \cup_\alpha \cup_\beta (\mathbf{x}_\alpha(U_\alpha) \times \mathbf{y}_\beta(V_\beta)) \\ &= \cup_\alpha \mathbf{x}_\alpha(U_\alpha) \times (\cup_\beta \mathbf{y}_\beta(V_\beta)) \\ &= \cup_\alpha \mathbf{x}_\alpha(U_\alpha) \times N \\ &= M \times N.\end{aligned}$$

Next, suppose that $\mathbf{z}_{\alpha\beta}(U_\alpha \times V_\beta) \cap \mathbf{z}_{\gamma\delta}(U_\gamma \times V_\delta) = W \neq \emptyset$. Note that the intersection takes the form

$$\begin{aligned}W &= (\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\gamma(U_\gamma)) \times (\mathbf{y}_\beta(V_\beta) \cap \mathbf{y}_\delta(V_\delta)) \\ &= W_{\alpha\gamma} \times W_{\beta\delta}.\end{aligned}$$

We have

$$\mathbf{z}_{\alpha\beta}^{-1}(W \cap \mathbf{z}_{\alpha\beta}(U_\alpha \times V_\beta)) = \mathbf{x}_\alpha^{-1}(W_{\alpha\gamma} \cap \mathbf{x}_\alpha(U_\alpha)) \times \mathbf{y}_\beta^{-1}(W_{\beta\delta} \cap \mathbf{y}_\beta(V_\beta))$$

and

$$\mathbf{z}_{\gamma\delta}^{-1}(W \cap \mathbf{z}_{\gamma\delta}(U_\gamma \times V_\delta)) = \mathbf{x}_\alpha^{-1}(W_{\alpha\gamma} \cap \mathbf{x}_\gamma(U_\gamma)) \times \mathbf{y}_\delta^{-1}(W_{\beta\delta} \cap \mathbf{y}_\delta(V_\delta))$$

Since the right-hand side in each of these identities is the Cartesian product of open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, the left-hand side is an open subset of $\mathbb{R}^{n+m}$. We wish to show that the map

$$\mathbf{z}_{\alpha\beta}^{-1} \circ \mathbf{z}_{\gamma\delta} : \mathbf{z}_{\gamma\delta}^{-1}(W \cap \mathbf{z}_{\gamma\delta}(U_\gamma \times V_\delta)) \to \mathbf{z}_{\alpha\beta}^{-1}(W \cap \mathbf{z}_{\alpha\beta}(U_\alpha \times V_\beta))$$

is smooth. It is easy to compute that for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$(\mathbf{z}_{\alpha\beta}^{-1} \circ \mathbf{z}_{\gamma\delta})(x, y) = ((\mathbf{x}_\alpha^{-1} \circ \mathbf{x}_\gamma)(x), (\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\delta)(y))$$

which shows that the map is smooth.
Now consider the projection $\pi_1 : M \times N \to M$ defined by $\pi_1(p, q) = p$. To show that π_1 is C^∞ it suffices to show that for local coordinate charts $\mathbf{x}$ and $\mathbf{z}$, the map $\mathbf{x}^{-1} \circ \pi_1 \circ \mathbf{z}$ is smooth. We compute

$$\begin{aligned}
(\mathbf{x}^{-1} \circ \pi_1 \circ \mathbf{z})(x, y) &= (\mathbf{x}^{-1} \circ \pi_1)(\mathbf{x}(x), \mathbf{y}(y)) \\
&= \mathbf{x}^{-1}(\mathbf{x}(x)) \\
&= x
\end{aligned}$$

which is the identity map and therefore smooth. A similar computation for the map $\pi_2 : M \times N \to N$ defined by $\pi_2(p, q) = q$ shows that the map $\mathbf{y}^{-1} \circ \pi_2 \circ \mathbf{z}$ is also the identity map, hence π_2 is a smooth mapping.