Math 641
Homework 4
Solutions

1. (4 points) Throughout this exercise we regard S^1 as an embedded submanifold of $\mathbb{R}^2$ given by the set of (x, y) with $x^2 + y^2 = 1$. First, let us show that S^1 is diffeomorphic to $\mathbb{R}/\mathbb{Z}$. Define a mapping $f : \mathbb{R}/\mathbb{Z} \to S^1$ by
$$f([x]) = (\cos(2\pi x), \sin(2\pi x))$$
where x is any representative of $[x]$. This map is well-defined by the periodicity of the cosine and sine functions, maps onto S^1, and is injective since
$$(\cos(2\pi x), \sin(2\pi x)) = (\cos(2\pi x'), \sin(2\pi x'))$$
if and only if $[x] = [x']$. The projection $\pi : x \to [x]$ is a local diffeomorphism while S^1 admits coordinate charts $(\mathbb{R}, \mathbf{x}_1)$ and $(\mathbb{R}, \mathbf{x}_2)$ where
$$\mathbf{x}_1^{-1}(x, y) = \frac{x}{1 + y}$$
and
$$\mathbf{x}_2^{-1}(x, y) = \frac{x}{1 - y}.$$
We compute
$$(\mathbf{x}_1^{-1} \circ f \circ \pi)(x) = \frac{\cos(2\pi x)}{1 + \sin(2\pi x)}$$
$$(\mathbf{x}_2^{-1} \circ f \circ \pi)(x) = \frac{\cos(2\pi x)}{1 - \sin(2\pi x)}$$
which are smooth maps. It remains to show that f has a smooth inverse. Recall that the principle branch of the arc cosine function has range in $[0, \pi]$. For $(x, y) \in S^1$ define
$$g(x, y) = \frac{1}{2\pi} \begin{cases} \arccos(x) & y \geq 0 \\ \pi + \arccos(-x) & y < 0 \end{cases}$$
and let
$$f^{-1}(x, y) = [g(x, y)]$$

To see that f^{-1} is smooth it suffices to consider the behavior near $(-1,0)$ and $(1,0)$. First consider the coordinate map $\gamma_1 : (0,2\pi) \to S^1$ given by $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$ and compute

$$(g \circ \gamma)(t) = \frac{t}{2\pi}$$

which shows in particular that g is smooth at $\gamma_1(\pi) = (-1,0)$. Next consider $\gamma_2(t) : (-\pi,\pi) \to S^1$ of the same form and note that

$$(g \circ \gamma_2)(t) = \begin{cases} \dfrac{1}{2\pi}(t+2\pi) & -\pi < t < 0 \\ \dfrac{1}{2\pi}t & 0 \le t < \pi \end{cases}.$$

Since $\left[\frac{t}{2\pi}+1\right] = \left[\frac{t}{2\pi}\right]$ it follows that $\left(f^{-1}\circ\gamma_2\right)(t) = [t/2\pi]$ so that f^{-1} is smooth at $\gamma_2(0) = (1,0)$. Hence f^{-1} is smooth.
Now we turn to the proof that $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is diffeomorphic to $S^1 \times \cdots \times S^1$ (n-fold Cartesian product). First, we observe that $S^1 \times \cdots \times S^n$ is diffeomorphic to $(\mathbb{R}/\mathbb{Z}) \times \cdots \times (\mathbb{R}/\mathbb{Z})$ since the mapping

$$([x_1], \cdots, [x_n]) \mapsto ((\cos(2\pi x_1), \sin(2\pi x_1)), \cdots, ((\cos(2\pi x_1), \sin(2\pi x_1)))$$

defines a diffeomorphism of these two product manifolds. It suffices, therefore, to show that T^n is diffeomorphic to $(\mathbb{R}/\mathbb{Z}) \times \cdots \times (\mathbb{R}/\mathbb{Z})$ (n-fold Cartesian product). Consider the mapping

$$\varphi : ([x_1], \cdots, [x_n]) \to [x_1, \cdots, x_n].$$

It is easy to see that this map is well-defined and onto. To check smoothness we use the fact that the projections $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and $\pi_n : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ give local diffeomorphisms, and setting $\pi^n = \pi \times \cdots \times \pi$ we see that $\pi_n^{-1} \circ \varphi \circ \pi^n$ takes the form

$$(x_1, \cdots, x_n) \mapsto (x_1 + k_1, \cdots, x_n + k_n)$$

for integers $k_1, \cdots, k_n$, and hence is smooth.

2. (2 points) If M has an atlas of charts of the form $\mathbf{x} : U \subset \mathbb{R}^n \to M$, the tangent bundle has an atlas of charts of the form

$$\mathbf{z} : U \times \mathbb{R}^n \to TM$$
$$(x,u) \to (\mathbf{x}(x), (d\mathbf{x})_x(u)).$$

If $\mathbf{z}$ and $\mathbf{w}$ are charts for TM with $\mathbf{z}(U\times\mathbb{R}^n)\cap\mathbf{w}(V\times\mathbb{R}^n) = W\times\mathbb{R}^n \neq \emptyset$, and if J_x is the Jacobian matrix of $(d\mathbf{x})_x$, the transition map

$$\mathbf{w}^{-1}\circ\mathbf{z} : \mathbf{z}^{-1}((W) \to \mathbf{w}^{-1}(W)$$

has block-diagonal Jacobian matrix of the form

$$J = \left[\begin{array}{cc} J_x & 0 \\ * & J_x \end{array}\right]$$

so that

$$\det(J) = \det(J_x)^2 > 0.$$

This shows that TM is orientable.

3. (4 points)

   (a) (2 points) We wish to show that a smooth surface S in $\mathbb{R}^3$ is orientable if and only if it admits a globally defined normal vector field. We begin with some local considerations Suppose that $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ is a local parameterization $(u,v) \mapsto \mathbf{x}(u,v)$. The vectors $\mathbf{v}_1 = (\partial\mathbf{x}/\partial u)(u,v)$ and $\mathbf{v}_2 = (\partial\mathbf{x}/\partial v)(u,v)$ form a basis for the tangent space $T_{\mathbf{x}(u,v)}S$ and the vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$ is a normal there. If $\mathbf{N}$ is a globally defined normal then

   $$\mathbf{N}\cdot\mathbf{n} = \mathbf{N}\cdot(\mathbf{v}_1\times\mathbf{v}_2)$$

   is positive if $\mathbf{N}$ and $\mathbf{n}$ are parallel, or negative if $\mathbf{N}$ and $\mathbf{n}$ are anti-parallel. The right-hand side is the determinant[1]

   $$\begin{vmatrix} v_{11} & v_{21} & N_1 \\ v_{12} & v_{22} & N_2 \\ v_{13} & v_{23} & N_3 \end{vmatrix}$$

   (the columns are the components of the vectors $\mathbf{N}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ respectively). Suppose that $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ is another coordinate system and that $\mathbf{x}(u,v) = \mathbf{y}(u',v')$. Then $\mathbf{w}_1 = (\partial\mathbf{y}/\partial u)(u',v')$

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[1] The vector triple product is usually defined as the determinant of the transpose of the matrix given here, but of course $\det(A) = \det(A^T)$ so this doesn't matter.

and $\mathbf{w}_2 = (\partial\mathbf{y}/\partial v)\,(u', v')$ also give a basis for the tangent space. From the identity

$$d\mathbf{x} = d\mathbf{y} \circ d\left(\mathbf{y}^{-1} \circ \mathbf{x}\right)$$

we see that if $d\left(\mathbf{y}^{-1} \circ \mathbf{x}\right)$ has Jacobian matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

then

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \\ v_{13} & v_{23} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \\ w_{13} & w_{23} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

so that

$$\begin{bmatrix} v_{11} & v_{21} & N_1 \\ v_{12} & v_{22} & N_2 \\ v_{13} & v_{23} & N_3 \end{bmatrix} = \begin{bmatrix} w_{11} & w_{21} & N_1 \\ w_{12} & w_{22} & N_2 \\ w_{13} & w_{23} & N_3 \end{bmatrix} \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and therefore

$$\begin{vmatrix} v_{11} & v_{21} & N_1 \\ v_{12} & v_{22} & N_2 \\ v_{13} & v_{23} & N_3 \end{vmatrix} = \begin{vmatrix} w_{11} & w_{21} & N_1 \\ w_{12} & w_{22} & N_2 \\ w_{13} & w_{23} & N_3 \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}. \tag{1}$$

Now suppose that S admits a globally defined normal field. Given any chart we can always permute coordinates so that $\mathbf{N}\cdot(\mathbf{v}_1 \times \mathbf{v}_2)$ is positive. Taking an atlas for S and modifying the coordinate maps as needed we can arrange that if $\mathbf{x}$ and $\mathbf{y}$ are any two coordinate maps so that $\mathbf{x}(q) = \mathbf{y}(s)$, we have that $\mathbf{N}\cdot(\mathbf{v}_1 \times \mathbf{v}_2)$ and $\mathbf{N}\cdot(\mathbf{w}_1 \times \mathbf{w}_2)$ are both positive. It follows from (1) that the Jacobian of the transition map $\mathbf{y}^{-1} \circ \mathbf{x}$ at q is positive. Thus we have constructed an atlas of charts for S with the property that all transition maps have positive Jacobian.

Suppose on the other hand that S admits such an atlas. In each coordinate patch we define a unit normal vector field by

$$\mathbf{N} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}.$$

To see that the definition is globally consistent we consider the unit normal fields given by two overlapping patches. Since the Jacobian determinant of the transition map is positive it follows again from (1) that the unit normals generated by the respective coordinate maps are parallel, and so coincide.

(b) (2 points) Consider the Möbius strip S viewed as an embedded surface. Consider the chart

$$\mathbf{x} : (0, 2\pi) \times (-1, 1) \to S$$
$$(u, v) \mapsto \begin{bmatrix} \left(1 + v\sin\left(\frac{u}{2}\right)\right)\cos u \\ \left(1 + v\sin\left(\frac{u}{2}\right)\right)\sin u \\ v\cos\left(\frac{u}{2}\right) \end{bmatrix}$$

It is not difficult to compute (or to reason geometrically) that a unit normal field is given by

$$\mathbf{N}_{\mathbf{x}(u,v)} = \begin{bmatrix} \cos\left(\frac{u}{2}\right)\sin u \\ \cos\left(\frac{u}{2}\right)\cos u \\ \sin\left(\frac{u}{2}\right) \end{bmatrix}.$$

However,

$$\lim_{(u,v)\to(0,0)} \mathbf{N}_{\mathbf{x}(u,v)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

while

$$\lim_{(u,v)\to(2\pi,0)} \mathbf{N}_{\mathbf{x}(u,v)} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

so that $\mathbf{N}$ does not have a continuous extension to S. Since any globally defined normal vector field must coincide with either $\mathbf{N}$ or $-\mathbf{N}$, this shows that S is not orientable.