# Math 641
# Homework 5
# Solutions

1. (a) We recall the formula for expansion by minors:

$$\det(A) = \sum_{j=1}^{n} a_{kj}(-1)^{k+j} M_{kj}$$

where $M_{ij}$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained from $A$ by deleting the $i$th row and $j$th column. Note that, in the notation of the problem, $a_{ij} = (a_i)_j$. To find $\partial(\det A)/\partial a_{ij}$, let

$$A(t) = A + te_{ij}$$

where $e_{ij}$ is the matrix with $a_{ij} = 1$ and all other entries zero. It is easy to see that

$$\det(A)(t) = \det A + t(-1)^{i+j} M_{ij}$$

so

$$\frac{\partial}{\partial a_{ij}}(\det A) = (-1)^{i+j} M_{ij}.$$

Now let $x = (x_1, \cdots, x_n)$ be a tangent vector in $T_A(\mathbb{R}^n \times \cdots \times \mathbb{R}^n)$ and set $(x_i)_j = x_{ij}$. Then

$$\begin{aligned} d(\det A)(x) &= \sum_{i,j=1}^{n} x_{ij}\frac{\partial}{\partial a_{ij}}(\det A) \\ &= \sum_{i,j=1}^{n} (-1)^{i+j} x_{ij} M_{ij} \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}(-1)^{i+j}(x_i)_j M_{ij}\right) \\ & \sum_{i=1}^{n} \det\begin{bmatrix} a_1 \\ \vdots \\ x_i \\ \vdots \\ a_n \end{bmatrix} \end{aligned}$$

as claimed.

(b) It suffices to show that $d(\det)$ has full rank (i.e., rank one) for each $A \in SL_n(\mathbb{R})$, and to do this it suffices to exhibit $x = (x_1, \cdots, x_n)$

with $d(\det A)(x) \neq 0$. For $A = (a_1, \ldots, a_n)$ let $x = (a_1, \ldots, a_n)$ and compute

$$\begin{aligned} d\left(\det\right)_A (a_1, \ldots, a_n) &= \sum_{i=1}^{n} \det \begin{bmatrix} e_1 \\ \vdots \\ a_i \\ \vdots \\ e_n \end{bmatrix} \\ &= n \det A \\ &= n. \end{aligned}$$

It now follows from the Implicit Function Theorem that $SL_n(\mathbb{R})$ is a smooth submanifold of codimension one, hence of dimension $n^2 - 1$.

2. .Let $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $R(x) = -x$. Then $R$ is an isometry of $\mathbb{R}^{n+1}$ since for any $u, v \in T_x\mathbb{R}^{n+1}$,

$$\begin{aligned} \langle dR_x(u), dR_x(v)\rangle_{T_{-x}\mathbb{R}^{n+1}} &= \langle -u, -v\rangle_{T_{-x}\mathbb{R}^{n+1}} \\ &= \langle u, v\rangle_{T_{-x}\mathbb{R}^{n+1}} \\ &= \langle u, v\rangle_{T_x\mathbb{R}^{n+1}}. \end{aligned}$$

The antipodal map on $S^n$ is a restriction of $R$ to $S^n$ and so is also an isometry. To see this, note that the identification map $i : S^n \to \mathbb{R}^{n+1}$ is an isometry (by definition of the induced metric!) and the restriction of $R$ to $S^n$ is given by $R_0 = i^{-1} \circ R \circ i$. The induced metric is given, for $p \in S^n$, by

$$\langle u, v\rangle_p = \langle (di)(u), (di)(v)\rangle_{i(p)}.$$

If $p \in S^n$ and $u, v \in T_pS^n$ then

$$\begin{aligned} \langle dR_0(u), dR_0(v)\rangle_{R_0(p)} &= \left\langle \left((di)^{-1} \circ R \circ di\right)(u), \left((di)^{-1} \circ R \circ di\right)(v)\right\rangle_{R_0(p)} \\ &= \langle (R \circ di)(u), (R \circ di)(v)\rangle_{R(p)} \\ &= \langle di(u), di(v)\rangle_{i(p)} \\ &= \langle u, v\rangle_p \end{aligned}$$

which shows that $R_0$ is an isometry of $S^n$.
Now suppose that $p \in P(\mathbb{R}^n)$ and let $\pi : S^n \to P(\mathbb{R}^n)$ be the canonical projection. The map $\pi$ is a local diffeomorphism so that if $q \in \pi^{-1}(p)$ then $d\pi_q$ is an isomorphism of vector spaces. We wish to define, for $u, v \in T_pP(\mathbb{R}^n)$,

$$\langle u, v\rangle_p = \left\langle d\pi^{-1}(u), d\pi^{-1}(v)\right\rangle_q$$

and we need to show that the definition is independent of the choice of $q$. Suppose that $\pi^{-1}(p) = \{q_1, q_2\}$ and define mappings $\pi_1 : U_1 \to P(\mathbb{R}^n)$

and $\pi_2 : U_2 \to P(\mathbb{R}^n)$ where $U_1$ is a neighborhood of $q_1$ in $S^n$, $U_2$ is a neighborhood of $q_2$ in $S^n$, and the neighborhoods are so chosen that $\pi_1$ and $\pi_2$ are local diffeomorphisms. Then $\pi_1^{-1} \circ \pi_2 = R_0$ on $\pi_2^{-1}(U_2 \cap \pi(U_1))$ and $\pi_1^{-1} \circ \pi_2(q_2) = R_0(q) = q_1$. We compute, for $u$ and $v$ in $T_pP(\mathbb{R}^n)$,

$$\begin{aligned}
\left\langle d\pi_1^{-1}(u), d\pi_1^{-1}(v)\right\rangle_{q_1} &= \left\langle d\left(\pi_1^{-1} \circ \pi_2\right) \circ d\pi_2^{-1}(u), d\left(\pi_1^{-1} \circ \pi_2\right) \circ d\pi_2^{-1}(v)\right\rangle_{q_1} \\
&= \left\langle dR_0 \circ d\pi_2^{-1}(u), dR_0 \circ d\pi_2^{-1}(v)\right\rangle_{q_1} \\
&= \left\langle d\pi_2^{-1}(u), d\pi_2^{-1}(v)\right\rangle_{q_2}
\end{aligned}$$

so that the definition is indeed independent of the choice of representative in $\pi^{-1}(p)$.

Finally we show that the projection map $\pi$ is a local isometry. We compute, for $q \in S^n$ and $u$ and $v$ in $T_qS^n$,

$$\begin{aligned}
\langle d\pi(u), d\pi(v)\rangle_{\pi(q)} &= \left\langle d\pi^{-1}(d\pi(u)), d\pi^{-1}(d\pi(v))\right\rangle_q \\
&= \langle u, v\rangle_q
\end{aligned}$$

(the first equality follows from the defintion of the metric on $P(\mathbb{R}^n)$). This shows that $\pi$ is a local isometry.

3. We'll use the global coordinate map

$$\mathbf{x} : (x, y) \to g_{x,y}$$

to parametrize $g$, where

$$g_{x,y} = \{t \mapsto yt + x\}\,.$$

First we compute the group law for $G$. Since

$$\begin{aligned}
g_{(x,y)} \circ g_{(x',y;)} &= t \mapsto y\left(y't + x'\right) + x \\
&= t \mapsto yy't + (yx' + x)
\end{aligned}$$

the law of composition on $G$ (viewed in the global coordinates) is

$$(x, y) \circ (x', y') = (yx' + x, yy')\,.$$

The identity element is $(0, 1)$ since

$$(x, y) \circ (0, 1) = (x, y)$$

The inverse of an element $(x, y)$ is then

$$(x, y)^{-1} = \left(-x/y, y^{-1}\right).$$

Note that

$$\begin{aligned}
(x, y)\,, (x', y') &\mapsto (yx' + x, yy') \\
(x, y) &\mapsto \left(-x/y, y^{-1}\right)
\end{aligned}$$

define smooth maps. Thus left translation is given by

$$\left(\mathbf{x}^{-1} \circ L_{(x',y')} \circ \mathbf{x}\right)(x,y) = (y'x + x', y'y)$$

and the Jacobian matrix of the map $\left(\mathbf{x}^{-1} \circ L_{(x',y')} \circ \mathbf{x}\right)$ at $(x,y)$ is given by

$$d\left(\mathbf{x}^{-1} \circ L_{(x',y')} \circ \mathbf{x}\right)_{(x',y')}(u,v) = \begin{pmatrix} y' & 0 \\ 0 & y' \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

(a) In what follows we let $\partial/\partial x|_{(x,y)} = d\mathbf{x}(e_1)$ and $\partial/\partial y|_{(x,y)} = d\mathbf{x}(e_2)$. For

$$\begin{aligned} u &= u_1 \left.\frac{\partial}{\partial x}\right|_{(0,1)} + u_2 \left.\frac{\partial}{\partial y}\right|_{(0,1)} \\ v &= v_1 \left.\frac{\partial}{\partial x}\right|_{(0,1)} + v_2 \left.\frac{\partial}{\partial y}\right|_{(0,1)} \end{aligned}$$

let us set $\langle u,\ v\rangle_e = u_1v_1 + u_2v_2$. We define a left-invariant metric by setting

$$\langle u, v\rangle_{\mathbf{x}(x,y)} = \left\langle dL_{\mathbf{x}(x,y)^{-1}}(u), dL_{\mathbf{x}(x,y)^{-1}}(v)\right\rangle_e.$$

If now

$$u = u_1 \left.\frac{\partial}{\partial x}\right|_{(x,y)} + u_2 \left.\frac{\partial}{\partial y}\right|_{(x,y)} \tag{1}$$

$$v = v_1 \left.\frac{\partial}{\partial x}\right|_{(x,y)} + v_2 \left.\frac{\partial}{\partial y}\right|_{(x,y)} \tag{2}$$

we have

$$\langle u, v\rangle_{\mathbf{x}(x,y)} = \left\langle dL_{\mathbf{x}(x,y)^{-1}}(u), dL_{\mathbf{x}(x,y)^{-1}}(v)\right\rangle_e$$

so it suffices to compute

$$d\left(\mathbf{x}^{-1} \circ L_{\mathbf{x}(x,y)^{-1}} \circ \mathbf{x}\right) = d\left(\mathbf{x}^{-1} \circ L_{\mathbf{x}(-x/y,y^{-1})} \circ \mathbf{x}\right)$$

which is a linear mapping with matrix

$$\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}$$

so that if $u$ is given by (1) then

$$dL_{\mathbf{x}(x,y)^{-1}}(u) = y^{-1}\left(u_1 \left.\frac{\partial}{\partial x}\right|_{(0,1)} + u_2 \left.\frac{\partial}{\partial y}\right|_{(0,1)}\right)$$

and similarly

$$dL_{\mathbf{x}(x,y)^{-1}}(v) = y^{-1}\left(v_1 \left.\frac{\partial}{\partial x}\right|_{(0,1)} + v_2 \left.\frac{\partial}{\partial y}\right|_{(0,1)}\right)$$

so that

$$\langle u, v \rangle_{\mathbf{x}(x,y)} = y^{-2} \left( u_1 v_1 + u_2 v_2 \right).$$

Observe that this quadratic form is invariant under orthogonal transformations (rotations) in the coordinates..

(b) Suppose that $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map that preserves the upper half-plane. Then $h$ induces a smooth map $H$ from $G$ to itself by

$$H = \mathbf{x} \circ h \circ \mathbf{x}^{-1}$$

and the matrix of $dH_{\mathbf{x}(x,y)}$ with respect to the coordinate basis is the Jacobian matrix of the mapping $h$. If $h$ is defined by a holomorphic mapping $f$ of the complex plane, i.e.,

$$h(x,y) = \left( \begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right)$$

with $z = x + iy$, and $f(x+iy) = u(x,y) + iv(x,y)$, the Jacobian matrix of $h$ is

$$\begin{aligned} \left( \begin{array}{cc} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{array} \right) &= \left( \begin{array}{cc} u_x(x,y) & u_y(x,y) \\ -u_y(x,y) & u_x(x,y) \end{array} \right) \\ &= \sqrt{u_x(x,y)^2 + u_y(x,y)^2} \left( \begin{array}{cc} \cos\theta(x,y) & \sin\theta(x,y) \\ -\sin\theta(x,y) & \cos\theta(x,y) \end{array} \right) \\ &= \left| f'(z) \right| R_\theta \end{aligned}$$

where in the first step we used the Cauchy-Riemann equations, in the second step we set

$$\begin{aligned} \cos\theta(x,y) &= \frac{u_x(x,y)}{\sqrt{u_x(x,y)^2 + u_y(x,y)^2}} \\ \sin\theta(x,y) &= \frac{u_y(x,y)}{\sqrt{u_x(x,y)^2 + u_y(x,y)^2}} \end{aligned}$$

and in the last step $R_\theta$ is the rotation matrix on the second line. It follows that, for any such map $f$ that preserves the upper half-plane, any point $p = \mathbf{x}(x,y)$ in $G$, and any tangent vectors $u$ and $v$ belonging to $T_pG$,

$$\langle dh(u), dh(v) \rangle_{h(\mathbf{x}(x,y))} = \langle dh(u), dh(v) \rangle_{\mathbf{x}(H(x,y))}$$

The matrix of $dh$ with respect to the coordinate vector fields is the Jacobian matrix of $H$ (which is a rotation and a dilation by $|f'(z)|$ so that if $u$ and $v$ are given by (1) and (2) we have

$$\langle dh(u), dh(v) \rangle_{\mathbf{x}(H(x,y))} = \frac{\left| f'(z) \right|^2 \left( u_1 v_1 + u_2 v_2 \right)}{\left( \operatorname{Im} f(z) \right)^2}$$

if $z = x + iy$.
Now consider the map

$$f(z) = \frac{az+b}{cz+d} \tag{3}$$

where $a, b, c, d$ are real numbers with $ad - bc = 1$ and note that

$$|f'(z)|^2 = \frac{1}{|cz+d|^2}$$

while

$$\operatorname{Im} f(z) = \frac{\operatorname{Im}(z)}{|cz+d|^2}.$$

It follows that if $h$ is induced by (3) then

$$\frac{|f'(z)|^2}{(\operatorname{Im} f(z))^2} = \frac{1}{\operatorname{Im}(z)^2}$$

and hence

$$\begin{aligned}\langle dh(u), dh(v)\rangle_{h(\mathbf{x}(x,y))} &= \frac{1}{\operatorname{Im}(z)^2}(u_1v_1 + u_2v_2) \\ &= \langle u, v\rangle_{\mathbf{x}(x,y)}\end{aligned}$$

which shows that $h$ is an isometry.