Math 641
Homework 6
Solutions

1. :Let us denote by $P_{c,t,t_0} : T_{c(t_0)}M \to T_{c(t)}M$ the parallel transport of a vector $v \in T_{c(t_0)}M$ along c to $T_{c(t)}M$. For the moment we'll denote this mapping by P (to conserve subscripts) and set $p = c(t_0)$, $q = c(t)$.. We wish to show that P is an isometry from T_pM to T_qM, and, if M is oriented, that P is an orientation-preserving map. Let $\{v_i\}_{i=1}^n$ be an orthonormal basis for T_pM and let $w_i = Pv_i$. Then $\{w_i\}_{i=1}^n$ is an orthonormal basis for T_qM and if $X = \sum \alpha_i v_i$, $Y = \sum \beta_i v_i$ belong to T_pM, then
$$\langle PX, PY\rangle_q = \sum_{i=1}^n \alpha_i\beta_i = \langle X, Y\rangle_p$$
so P is an isometry.
Now suppose that M has a given orientation, i.e., there is an atlas for M consists of charts whose transition maps have positive determinant. In each chart $\mathbf{x} : U \to M$, the coordinate vector fields $\left\{ \partial/\partial x_i|_p \right\}_{i=1}^n$ at $p \in \mathbf{x}(U)$ are taken to be a positively oriented basis. We wish to show that $P(t) := P_{c,t_0,t}$ is an orientation preserving map of $T_{c(t_0)}M$ onto $T_{c(t)}M$ for each t. We note that if $t_1 < t_2$
$$P_{c,t_0,t_2} = P_{c,t_1,t_2} \circ P_{c,t_0,t_1}. \tag{1}$$
Set $v_i = \partial/\partial x_i|_{c(t_0)}$ and $X_i(t) = P(t)v_i$. It suffices to show that for all t, $\{X_i(t)\}_{i=1}^n$ is a properly oriented basis for $T_{c(t)}M$. Since $P(0) = I$ it is easy to see that the change of basis matrix from the basis $\{X_i(t)\}_{i=1}^n$ to the basis $\left\{ \partial/\partial x_i|_{c(t)} \right\}_{i=1}^n$ for t small is close to the identity, hence has positive determinant. By repeating this argument we can show that $\{X_i(t)\}_{i=1}^n$ is properly oriented for t in any closed interval I with the property that $t_0 \in I$ and $c(I)$ is contained in $\mathbf{x}(U)$. Suppose that $(\mathbf{y}, V)$ is a coordinate chart that overlaps $(\mathbf{x}, U)$ and $c(t_1) \in \mathbf{x}(U) \cap \mathbf{y}(V)$. The condition on overlapping charts shows that $\{X_i(t_1)\}_{i=1}^n$ has the same orientation as the two bases$\left\{ \partial/\partial x_i|_{c(t_1)} \right\}_{i=1}^n$ and $\left\{ \partial/\partial y_i|_{c(t_1)} \right\}_{i=1}^n$. Repeating the argument above taking $v_i = \{X_i(t_1)\}_{i=1}^n$ and replacing

$P_{c,t_0,t}$ by $P_{c,t_1,t}$ shows, in virtue of (**??**), that $\{X_i(t)\}_{i=1}^n$ is properly oriented for any t in $[t_1, t_2]$ if $c([t_1, t_2]) \in \mathbf{y}(V)$.

2. Suppose that X and Y are smooth vector fields on a Riemannian manifold M, let ∇ denote the Levi-Civita connection, and let P_{c,t,t_0} denote parallel transport along $c(t)$ from $T_{c(t_0)}M$ to $T_{c(t)}M$. Note that $P_{c,t,t_0} : T_{c(t_0)}M \to T_{c(t)}M$ is an isomorphism of vector spaces and is therefore an invertible map. The purpose of this problem is to prove the formula

$$(\nabla_X Y)(p) = \left.\frac{d}{dt}\right|_{t=t_0} \left(P^{-1}_{c,t,,t_0} Y(c(t))\right)$$

where $c(t)$ is an integral curve of X with $c(t_0) = p$, and, by definition $c'(t) = X(c(t))$. P^{-1}_{c,t,t_0} is the inverse of P_{c,t,t_0}. Stated in words, this formula says that the covariant derivative of Y in the X direction is the same as the rate of change of Y along the curve $c(t)$ computed by pulling the vector field $Y(c(t))$ back to the tangent space T_pM.

To prove the result, we choose an orthonormal basis $\{e_i\}_{i=1}^n$ for T_pM and define

$$X_i(t) = P_{c,t,t_0} e_i.$$

Thus, for each t, $\{X_i(t)\}_{i=1}^n$ is an orthonormal basis for $T_{c(t)}M$ and

$$\frac{DX_i(t)}{dt} = 0.$$

Now let us write

$$Y(c(t)) = \sum_j y^j(t) X_j(t)$$

and observe that

$$P^{-1}_{c,t,t_0} Y(c(t)) = \sum_j y^j(t) e_i$$

and hence

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(P^{-1}_{c,t,,t_0} Y(c(t))\right) = \sum_j \left(y^j\right)'(t_0) e_i.$$

On the other hand, we know that

$$
\begin{aligned}
(\nabla_X Y)(p) &= \left.\frac{D}{dt}\right|_{t=t_0} Y(c(t)) \\
&= \left.\frac{D}{dt}\right|_{t=t_0} \left(\sum_j y^j(t) X_j(t) \right) \\
&= \sum_j \left(y^j\right)' (t_0) X_j(t_0) \\
&= \sum_j \left(y^j\right)' (t_0) v_j
\end{aligned}
$$

which proves the desired inequality.

3. This problem is postponed to problem set 7.