Math 641
Homework 7
Due Monday, November 17

For the second problem you may, if you wish, use the Calculus of Variations to derive the equations of motion for a geodesic. From these equations you can deduce the connection coefficients.

1. do Carmo page 57, problem 4 revisited. Suppose that M^2 is a surface in $\mathbb{R}^3$ with local parameterization

$$\begin{aligned} \mathbf{x} : U \subset \mathbb{R}^2 &\to M^2 \\ (u, v) &\mapsto \mathbf{x}(u, v) \end{aligned}$$

We'll set

$$\mathbf{x}_u = \begin{pmatrix} \frac{\partial x_1}{\partial u} \\ \frac{\partial x_2}{\partial u} \\ \frac{\partial x_2}{\partial u} \end{pmatrix}, \quad \mathbf{x}_v = \begin{pmatrix} \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial v} \end{pmatrix}$$

(a) Show that the metric coefficients g_{ij} are given by

$$\begin{aligned} g_{11} &= \mathbf{x}_u \cdot \mathbf{x}_u \\ g_{12} &= \mathbf{x}_u \cdot \mathbf{x}_v \\ g_{22} &= \mathbf{x}_v \cdot \mathbf{x}_v \end{aligned}$$

(b) Recall that the connection coefficients are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(g_{mi,j} + g_{jm,i} - g_{ij,m} \right)$$

Let's define

$$\gamma^k_{ij} = \frac{1}{2} \left(g_{mi,j} + g_{jm,i} - g_{ij,m} \right).$$

Show that

$$\begin{aligned} \gamma_{11}^1 &= \mathbf{x}_{uu} \cdot \mathbf{x}_u, \quad \gamma_{11}^2 = \mathbf{x}_{uu} \cdot \mathbf{x}_v \\ \gamma_{12}^1 &= \mathbf{x}_{uv} \cdot \mathbf{x}_u, \quad \gamma_{12}^2 = \mathbf{x}_{uv} \cdot \mathbf{x}_v \\ \gamma_{22}^1 &= \mathbf{x}_{vv} \cdot \mathbf{x}_u, \quad \gamma_{22}^2 = \mathbf{x}_{vv} \cdot \mathbf{x}_v \end{aligned}$$

(c) Suppose that c is a curve in M with $\mathbf{x}^{-1} \circ c(t) = (u(t), v(t))$, and suppose that $V(t)$ is a vector field along c with

$$V(t) = v^1(t)\mathbf{x}_u(u(t), v(t)) + v^2(t)\mathbf{x}_v(u(t), v(t)).$$

Show that

$$\begin{aligned} \frac{dV}{dt}(t) = & \frac{dv^1}{dt}(t)\mathbf{x}_u + \frac{dv^2}{dt}(t)\mathbf{x}_v \\ & + v^1(t)\left(\mathbf{x}_{uu}\dot{u} + \mathbf{x}_{uv}\dot{v}\right) \\ & + v^2(t)\left(\mathbf{x}_{vu}\dot{u} + \mathbf{x}_{vv}\dot{v}\right) \end{aligned}$$

(d) Show that $V(t)$ is parallel if and only if dV/dt is orthogonal to $T_{c(t)}M$. *Hint*: dV/dt is orthogonal to $T_{c(t)}M$ if and only if the two equations

$$\begin{aligned} \mathbf{x}_u \cdot \frac{dV}{dt} &= 0 \\ \mathbf{x}_v \cdot \frac{dV}{dt} &= 0 \end{aligned}$$

hold. Use these equations togther with the result of part (b) to show that dV/dt is orthogonal to $T_{c(t)}M$ if and only if the the components v^1 and v^2 obey the equations of parallel transport:

$$\begin{aligned} \frac{dv^1}{dt} + \sum_{ij} \Gamma_{ij}^1 \frac{dx^i}{dt} v^j &= 0 \\ \frac{dv^2}{dt} + \sum_{ij} \Gamma_{ij}^1 \frac{dx^2}{dt} v^j &= 0 \end{aligned}$$

Here $dx^1/dt = \dot{u}$ and $dx^2/dt = \dot{v}$. It suffices (why?) to show that

$$\sum_k g_{ik} \frac{dv^k}{dt} + \sum \gamma_{jk}^i \frac{dx^j}{dt} v^k = 0$$

for $i = 1, 2$.

2. do Carmo, page 57, problem 8 (connection for the Poincaré metric)