Math 641
Homework 7
Due Monday, November 17

1. On $\mathbf{R}^2_+ = \{(x,y) : y > 0\}$ we consider the metric given by $g_{11} = g_{22} = y^{-2}$.

   (a) We'll use the fact that the geodesic equations can be derived from the energy functional

   $$E(\gamma) = \frac{1}{2}\int_a^b \frac{1}{y^2}\left(\dot{x}^2 + \dot{y}^2\right)\, dt.$$

   Taking

   $$F(x,y,\dot{x},\dot{y}) = \frac{1}{2y^2}(\dot{x}^2 + \dot{y}^2)$$

   we have

   $$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = -y^{-3}(\dot{x}^2 + \dot{y}^2)$$
   $$\frac{\partial F}{\partial \dot{x}} = \frac{\dot{x}}{y^2}, \quad \frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{y^2}$$

   so that the Euler-Lagrange equations become

   $$0 = \frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right)$$
   $$-y^{-3}(\dot{x}^2 + \dot{y}^2) = \frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right)$$

   or

   $$\ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0$$
   $$\ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0$$

   from which we read off that

   $$\Gamma^1_{11} = 0, \quad \Gamma_{12} = -\frac{1}{y}, \quad \Gamma^1_{22} = 0$$
   $$\Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^2_{12} = 0, \quad \Gamma^2_{22} = -\frac{1}{y}.$$

(b) Let $v_0 = (0,1)$ be a tangent vector at $(0,1)$. Let $v(t)$ be the parallel transport of v_0 along the curve $(x(t), y(t)) = (t, 1)$. Note that the Riemannian metric is identical to the Euclidean metric along the line $y = 1$ so that $v(t)$ is a (Euclidean) unit vector. If

$$v(t) = v^1(t)\frac{\partial}{\partial x} + v^2(t)\frac{\partial}{\partial y}$$

then

$$\nabla_{dc/dt} v(t) = \frac{dv^1}{dt}\frac{\partial}{\partial x} + \frac{dv^2}{dt}\frac{\partial}{\partial y} + v^1 \nabla_{dc/dt}\left(\frac{\partial}{\partial x}\right) + v^2 \nabla_{dc/dt}\left(\frac{\partial}{\partial y}\right).$$

Note that

$$\begin{aligned} \nabla_{dc/dt}\left(\frac{\partial}{\partial x}\right) &= \nabla_{\partial/\partial x}\left(\frac{\partial}{\partial x}\right) \\ &= \Gamma^1_{11}\frac{\partial}{\partial x} + \Gamma^2_{11}\frac{\partial}{\partial y} \\ &= \frac{1}{y}\frac{\partial}{\partial y} \end{aligned}$$

and

$$\begin{aligned} \nabla_{dc/dt}\left(\frac{\partial}{\partial y}\right) &= \nabla_{\partial/\partial x}\left(\frac{\partial}{\partial y}\right) \\ &= \Gamma^1_{12}\frac{\partial}{\partial x} + \Gamma^2_{12}\frac{\partial}{\partial y} \\ &= -\frac{1}{y}\frac{\partial}{\partial x} \end{aligned}$$

Hence, if $\nabla_{dc/dt} v(t) = 0$ we get

$$\begin{aligned} \frac{dv^1}{dt} - \frac{1}{y}v^2 &= 0 \\ \frac{dv^2}{dt} + \frac{1}{y}v^2 &= 0 \end{aligned}$$

Along the line $y = 1$ we get

$$\begin{aligned} \frac{dv^1}{dt} &= v^2 \\ \frac{dv^2}{dt} &= -v^1 \end{aligned}$$

If we set $v^1(t) = \cos\theta(t)$, $v^2(t) = \sin\theta(t)$ it follows that $\dot{\theta}(t) = -1$ and as $\theta(0) = \pi/2$ we recover $\theta(t) = \pi 2 - t$.

2. This problem concerns geodesics of a surface of revolution in $\mathbf{R}^3$. Suppose (u, v) are Cartesian coordinates in $\mathbf{R}^2$ and

$$U = \{(u, v) : u_0 < u < u_1,\ v_0 < v < v_1\}.$$

Let $\varphi : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ by

$$\varphi(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

where f and g are differentiable with $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$ The Jacobian matrix of φ is

$$J = \begin{pmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v) \end{pmatrix}$$

Note that the rank is always two since either (i) $g'(v) \neq 0$ and $f(v) \neq 0$ or (ii) $f'(v) \neq 0$ and $f(v) \neq 0$, so that there is always a 2×2 submatrix of the Jacobian that is nonzero. The image $\varphi(U)$ is a rotation of the parametric curve $(f(v), g(v))$ in $\mathbf{R}^2$ about the z-axis. *Meridians* are curves $\varphi(u_0, v)$ and *parallels* are curves $\varphi(u, v_0)$.

(a) The induced metric is

$$\langle X, Y\rangle_{(u,v)} = \langle \varphi_*(X), \varphi_*(Y)\rangle$$

where the right-hand inner product is the usual inner product on $\mathbf{R}^3$. If J is the Jacobian matrix then the matrix of the inner product with respect to the usual basis of $T_{(u,v)}\mathbf{R}^2$ is

$$J^T J = \begin{pmatrix} f(v)^2 & 0 \\ 0 & h(v) \end{pmatrix}$$

where

$$h(v) = (f'(v))^2 + (g'(v))^2$$

so that

$$\begin{aligned} g_{11}(u, v) &= f(v)^2 \\ g_{12}(u, v) &= 0 = g_{21}(u, v) \\ g_{22}(u, v) &= h(v) = f'(v)^2 + g'(v)^2 \end{aligned}$$

(b) to find the equations of a geodesic we will consider the energy functional

$$E(\gamma) = \frac{1}{2}\int_a^b \left\{ f(v)^2 \dot{u}(t)^2 + h(v)\dot{v}(t)^2 \right\} dt$$

so that

$$F(u, v, \dot{u}, \dot{v}) = \frac{1}{2}\left\{ f(v)^2 \dot{u}^2 + h(v)\dot{v}^2 \right\}$$

Here

$$\begin{aligned} \frac{\partial F}{\partial u} &= 0, \quad \frac{\partial F}{\partial v} = f(v)f'(v)\dot{u}^2 + \frac{1}{2}h'(v)\dot{v}^2 \\ \frac{\partial F}{\partial \dot{u}} &= f(v)^2\dot{u}, \quad \frac{\partial F}{\partial \dot{v}} = h(v)\dot{v} \end{aligned}$$

so (writing f for $f(v)$, f' for $f'(v)$, etc.), the equations of motion are

$$\begin{aligned} 0 &= \frac{d}{dt}\left(f^2\dot{u}\right) \\ \left\{ ff'\dot{u}^2 + \frac{1}{2}h'\dot{v}^2 \right\} &= \frac{d}{dt}\left(h\dot{v}\right) \end{aligned}$$

or

$$\begin{aligned} \ddot{u} + \frac{2f'f''}{f^2}\dot{u}\dot{v} &= 0 \\ \ddot{v} + \frac{h'}{h}\dot{v}^2 - \frac{\left\{ ff'\dot{u}^2 + \frac{1}{2}h'\dot{v}^2 \right\}}{h} &= 0 \end{aligned}$$

Remembering the definition of h, we get

$$\ddot{u} + \frac{2f'f''}{f^2}\dot{u}\dot{v} = 0 \qquad (1)$$

$$\ddot{v} + \frac{f'f'' + g'g''}{f'^2 + g'^2}\dot{v}^2 - \frac{ff'\dot{u}^2}{f'^2 + g'^2} = 0 \qquad (2)$$

(c) The energy of a geodesic is

$$E(t) = \frac{1}{2}\left\{ f^2\dot{u}^2 + h\dot{v}^2 \right\}$$

so the time rate of change of energy is

$$
\begin{aligned}
\dot{E}(t) &= \frac{1}{2}\left\{2ff'\dot{v}\dot{u}^2 + 2f^2\dot{u}\ddot{u} + h'\dot{v}^3 + 2h\dot{v}\ddot{v}\right\} \\
&= \frac{1}{2}\left\{2ff'\dot{v}\dot{u}^2 - 4\dot{u}f'f''\dot{u}\dot{v} + h'\dot{v}^3 + 2h\dot{v}\ddot{v}\right\} \\
&= h\dot{v}\left\{\frac{ff'}{h}\dot{u}^2 - \frac{2f'f''}{h}\dot{u}^2 + \frac{h'}{2h}\dot{v}^2 + \ddot{v}\right\} \\
&= h\dot{v}\left\{\ddot{v} - \left(\frac{f'f''}{h}\dot{u}^2 - \frac{h'}{2h}\dot{v}^2\right)\right\}
\end{aligned}
$$

where we have used (2) to substitute for $\ddot{u}$ in the second line. The second factor on the last line is identically zero if (2) holds.
A parallel has tangent vector $\varphi_*(\partial/\partial u)$ so the angle made by $\gamma(t)$ with a parallel is given by

$$
\cos\beta = \frac{\langle \gamma'(t), \varphi_*(\partial/\partial u)\rangle}{\|\gamma'(t)\|\,\|\varphi_*(\partial/\partial u)\|} = \frac{f^2\dot{u}}{C\sqrt{f^2}}
$$

where $C = \|\gamma'(t)\|$ is a constant. Observe that f is the radius of the parallel so that

$$
r\cos\beta = \frac{f^2\dot{u}}{C}.
$$

The first geodesic equation says that

$$
\frac{d}{dt}\left(f^2\dot{u}\right) = 0
$$

which together with the fact that $\|\gamma'(t)\|$ is constant implies that $r\cos\beta$ is a constant.

(d) [to be done later!]