Math 641
Homework 8
Solutions

1. We assume given an orthonormal basis of vector fields $\{E_i\}_{i=1}^n$ so that $\nabla_{E_i} E_j(p) = 0$ for all i, j. To compute the gradient of f, defined by
$$\langle \operatorname{grad} f, v\rangle = df_p(v)$$
we compute
$$\begin{aligned}\langle \operatorname{grad} f, E_i\rangle &= df_p(E_i)\\ &= E_i(f)\end{aligned}$$
so that
$$\left(\operatorname{grad} f\right)(p) = \sum_{i=1}^n E_i(f) E_i(p)$$
If $X = \sum_{i=1}^n f_i E_i$ then
$$\begin{aligned}\operatorname{div} X &= \operatorname{tr}\left(Y \mapsto \nabla_Y X\right)\\ &= \sum_{j=1}^n \left\langle E_j, \nabla_{E_j} X\right\rangle\\ &= \sum_{j=1}^n \left\langle E_j, \nabla_{E_j}\left(\sum_{i=1}^n f_i E_i\right)\right\rangle\\ &= \sum_{j=1}^n \left\langle E_j, \left(\sum_{i=1}^n E_j(f_i) E_i\right)\right\rangle\\ &= \sum_{j=1}^n E_j(f_j)\end{aligned}$$
In Euclidean space the usual basis $\{e_i\}_{i=1}^n$ corresponding to coordinate vector fields $\{\partial/\partial x_i\}_{i=1}^n$ has the stated property since the connection vanishes. This gives the formulas
$$\left(\operatorname{grad} f\right)(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i$$
$$\operatorname{div}(X) = \sum_{i=1}^n \frac{\partial X_i}{\partial x_i}$$

if $X = \sum_{i=1}^{n} X_i e_i$.

2. We define

$$\Delta f = \operatorname{div} \operatorname{grad} f$$

(a) If E_i is a geodesic frame at $p \in M$, then

$$\Delta f(p) = \operatorname{div}\left(\sum E_i(f) E_i\right)$$

Since $\{E_i\}$ is a geodesic frame at p, $\nabla_{E_j} E_i(p) = 0$ and

$$\begin{aligned} \nabla_{E_j}\left(E_i(f)E_i\right)(p) &= \left(E_j E_i(f)(p)\right) E_i(p) + E_i(f) \nabla_{E_i} E_j(p) \\ &= \left(E_j E_i\left(f\right)\right)(p) E_i(p) \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{tr}\left(Y \mapsto \nabla_Y \left(\operatorname{grad} f\right)\right)(p) &= \sum_{j=1}^{n} \left\langle E_j(p), \nabla_{E_j}\left(\operatorname{grad}(f)\right)\right\rangle \\ &= \sum_{j=1}^{n} \left(E_j E_j f\right)(p) \end{aligned}$$

and hence

$$\Delta f(p) = \sum_{j=1}^{n} (E_j E_j f)(p).$$

In $\mathbb{R}^n$ the usual coordinate vector fields form a geodesic frame so it's immediate that

$$\Delta f(p) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(p).$$

(b) From the formula

$$\begin{aligned} E_i E_i (f \cdot g) &= E_i(f E_i g + g E_i f) \\ &= \left(E_i E_i f\right) g + 2\left(E_i f\right)\left(E_i g\right) + f\left(E_i E_i g\right) \end{aligned}$$

we get

$$\Delta\left(f \cdot g\right) = \left(\Delta f\right) g + 2\left\langle \nabla f, \nabla g\right\rangle + f\left(\Delta g\right)$$

at p. Since the resulting expressions are invariant we conclude that the formula holds globally.