Math 641
Midterm Answers

1. Consider the differentiable structures

$$\mathbf{x}_1(x) = x$$
$$\mathbf{x}_2(x) = x^3$$

on $M = \mathbf{R}$, where $x \in \mathbf{R}$.

(a) Let $i : (\mathbf{R}, \mathbf{x}_1) \to (\mathbf{R}, \mathbf{x}_2)$ be the identity map. Read through the charts we have $(\mathbf{x}_2^{-1} \circ i \circ \mathbf{x}_1)(x) = \mathbf{x}_2^{-1}(x) = x^{1/3}$ which is not differentiable at $x = 0$. It follows that the differentiable structures are distinct.

(b) If $f : (\mathbf{R}, \mathbf{x}_1) \to (\mathbf{R}, \mathbf{x}_2)$ is given by $f(x) = x^3$, then read through the charts we have $(\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1)(x) = \mathbf{x}_2^{-1}(x^3) = x$ which is a diffeomorphism. Hence $(\mathbf{R}, \mathbf{x}_1)$ and $(\mathbf{R}, \mathbf{x}_2)$ are diffeomorphic.

2. Let $F : \mathbf{R}^3 \to \mathbf{R}^4$ be given by

$$F(x, y, z) = \left(x^2 - y^2, xy, xz, yz\right)$$

for $p = (x, y, z) \in \mathbf{R}^3$. Let $S^2 \subset \mathbf{R}^3$ be the unit sphere $x^2 + y^2 + z^2 = 1$. Observe that $\varphi := F|_{S^2}$ obeys $\varphi(-p) = \varphi(p)$ and consider the mapping

$$\psi : P^2(\mathbf{R}) \to \mathbf{R}^4$$
$$[p] \mapsto \varphi(p)$$

where p is either representative of $[p]$.

(a) Let $\pi : S^2 \to P^2(\mathbf{R})$ be the natural projection, which is a local diffeomorphism, and let $i : S^2 \to \mathbf{R}^3$ be the natural embedding so that $\varphi = F \circ i$ and $\psi \circ \pi = F \circ i$. If $\mathbf{x} : U \to S^2$ is a coordinate chart, we need to show that the map $F \circ i \circ \mathbf{x}$ has a Jacobian of maximal rank. We'll work this out for the map $\mathbf{x}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$; it would suffice to consider this map and two others to get a covering of $P^2(\mathbf{R})$. We compute

$$(F \circ i \circ \mathbf{x})(u, v) = \left(u^2 - v^2, uv, u\sqrt{1 - u^2 - v^2}, v\sqrt{1 - u^2 - v^2}\right)$$

so that the Jacobian is

$$\begin{pmatrix} 2u & -2v \\ v & u \\ \sqrt{1-u^2-v^2} & \dfrac{-uv}{\sqrt{1-u^2-v^2}} \\ \dfrac{-uv}{\sqrt{1-u^2-v^2}} & \sqrt{1-u^2-v^2}\left(\dfrac{1-u^2-2v^2}{1-u^2-v^2}\right) \end{pmatrix}$$

The top 2×2 minor has determinant

$$2\left(u^2+v^2\right)$$

which is nonzero unless $(u, v) = (0, 0)$. In this case the bottom 2×2 minor has determinant 1, so the Jacobian has rank 2 for any (u, v) with $u^2 + v^2 < 1$. Similar considerations apply to the other coordinate maps.

(b) We wish to show that if $(x^2 - y^2, xy, xz, yz) = (x'^2 - y'^2, x'y', x'z', y'z')$ then $(x, y, z) = \pm(x', y', z')$. First observe that if $z = x + +iy$, $w = x' + iy'$, then we immediately recover $z^2 = w^2$, and hence $z = \pm w$, from the identity $z^2 = (x^2 - y^2) + 2ixy$. Either $z^2 = w^2 = 0$ and hence $z = \pm 1$ and $z' = \pm 1$, or at least one of x and y is nonzero. If say x is nonzero, we have $xz = x'z'$ which gives $z' = \pm z$ (with the same sign as in the relation $z = \pm w$). This shows the desired relation.

(c) To show that ψ is homeomorphic onto its range, we use the following fact (see, for example Rudin, *Principles of Mathematical Analysis*, Theorem 4.17 for the metric space version): if f is a continuous 1:1 map of a compact topological space X onto another topological space Y, the inverse map f^{-1} is continuous.[1] Differentiablily of ψ implies continuity, and we proved injectivity in (b).

3. First, suppose that $f(p) = \langle X(p), Y(p) \rangle_p$ is a smooth function for any $X, Y \in \mathcal{X}(M)$. Let $\mathbf{x} : U \to M$ be a coordinate chart and let $\partial/\partial x_i|_p$

[1] Proof: It suffices to show that for every open subset V of X, $f(V)$ is open in Y. Fix such a V. the complement V^c of V is closed in X, hence compact, so $f(V^c)$ is compact in Y, and so closed in Y. Since f is one-to-one and onto, $f(V)^c = f(V^c)$ is closed, so $f(V)$ is open as claimed. Hence f^{-1} is continuous.

be the coordinate vector fields. We can find vector fields X and Y on M so that $X(p) = \partial/\partial x_i|_p$ and $Y(p) = \partial/\partial x_i|_p$ in a neighborhood of given $q \in \mathbf{x}(U)$. It follows that $p \mapsto g_{ij}(p)$ is smooth near q.
Next, suppose that the functions g_{ij} are smooth in any local coordinates. If $X, Y \in \mathcal{X}(M)$ then

$$X(p) = \sum_{i=1}^{n} a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p$$
$$Y(p) = \sum_{j=1}^{n} b_j(p) \left.\frac{\partial}{\partial x_j}\right|_p$$

so that

$$\langle X(p), Y(p)\rangle_p = \sum_{i,j=1}^{n} g_{ij}(p) a_i(p) b_j(p).$$

Since $X, Y \in \mathcal{X}(M)$, the coefficients a_i and b_i are smooth functions, and hence the right-hand side defines a smooth function of p.