Math 641
Midterm Exam

**Instructions**: You should work on these problems alone. Please give neatly written, well-organized, and complete solutions to the following problems. Your written solutions are due at the beginning of class on Monday, October 26.

1. Consider the differentiable structures

$$\mathbf{x}_1(x) = x$$
$$\mathbf{x}_2(x) = x^3$$

on $M = \mathbf{R}$, where $x \in \mathbf{R}$. Show that

   (a) the identity mapping $i : (\mathbf{R}, \mathbf{x}_1) \to (\mathbf{R}, \mathbf{x}_2)$ is not a diffeomorphism, so that the maximal structures determined by $(\mathbf{R}, \mathbf{x}_1)$ and $(\mathbf{R}, \mathbf{x}_2)$ are distinct

   (b) the mapping $f : (\mathbf{R}, \mathbf{x}_1) \to (\mathbf{R}, \mathbf{x}_2)$ given by $f(x) = x^3$ is a diffeomorphism, so that these two smooth manifolds are diffeomorphic.

2. Let $F : \mathbf{R}^3 \to \mathbf{R}^4$ be given by

$$F(x, y, z) = \big(x^2 - y^2, xy, xz, yz\big)$$

for $p = (x, y, z) \in \mathbf{R}^3$. Let $S^2 \subset \mathbf{R}^3$ be the unit sphere $x^2 + y^2 + z^2 = 1$. Observe that $\varphi := F|_{S^2}$ obeys $\varphi(-p) = \varphi(p)$ and consider the mapping

$$\psi : P^2(\mathbf{R}) \to \mathbf{R}^4$$
$$[p] \mapsto \varphi(p)$$

where $p$ is either representative of $[p]$. Prove that:

   (a) $\psi$ is an immersion, i.e., $d\psi_q$ is injective for all $q \in P^2(\mathbf{R})$.

   (b) $\psi$ is injective, i.e., $\psi(q_1) = \psi(q_2)$ implies that $q_1 = q_2$.

   (c) $\psi$ is an embedding, i.e., $\psi$ is a homeomorphism onto its range. *Hint*: $P^2(\mathbf{R})$ is compact.

3. Let $M$ be a smooth manifold. Recall that if $\mathbf{x} : U \subset \mathbb{R}^n \to M$ is a coordinate chart on $M$, the coordinate vector fields $\partial/\partial x_i|_{\mathbf{x}(q)}$ are defined by

$$\left.\frac{\partial}{\partial x_i}\right|_{\mathbf{x}(q)} = d\mathbf{x}_q(e_i).$$

Recall that a vector field $X$ on $M$ is smooth if for any such coordinate chart, the coefficients $a_i$ in the expansion

$$X(p) = \sum_{i=1}^{n} a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p$$

are smooth functions of $p \in \mathbf{x}(U)$. Let $p \mapsto \langle \cdot , \cdot \rangle_p$ be a correspondence which assigns to each $p \in M$ an inner product $\langle \cdot , \cdot \rangle_p$ on $T_pM$. Prove that the following two smoothness conditions on the correspondence are equivalent:

(a) For all $C^\infty$ vector fields $X$ and $Y$ on $M$, the function $f(p) = \langle X(p), Y(p) \rangle_p$ is smooth.

(b) For any coordinate chart $\mathbf{x} : U \subset \mathbb{R}^n \to M$, and coordinate vector fields $\partial/\partial x_i|_p$, the functions

$$g_{ij}(p) = \left\langle \left.\frac{\partial}{\partial x_i}\right|_p , \left.\frac{\partial}{\partial x_i}\right|_p \right\rangle_p$$

defined on $\mathbf{x}(U)$ are smooth functions for $1 \leq i, j \leq n$.