1. Introduction

The purpose of these notes is to survey inverse scattering for the Schrödinger equation on the line. We will follow closely the discussion in Deift and Trubowitz’ classic paper [2].

Schrödinger’s equation for the wave function of a one-dimensional particle moving under the influence of a potential \( q(x) \) is

\[
-\frac{\hbar^2}{2m}\psi_{xx}(x, t) + q(x)\psi(x, t) = i\hbar\psi_t(x, t)
\]

We will always assume that \( q \) is a smooth, rapidly decreasing function, say \( q \in \mathcal{S}(\mathbb{R}) \), the space of smooth functions of rapid decrease.\(^1\) If we look for solutions with fixed energy of the form

\[
\psi(x, t) = \exp(-itE/\hbar)y(x)
\]
then the function \( y \) obeys the differential equation
\[
-\frac{\hbar^2}{2m} y_{xx} + q(x)y = Ey(x)
\]
which is an eigenvalue problem for the Schrödinger operator
\[
L_q = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + q(x)
\]
We will choose “natural units” in which \( \hbar = m = 1 \), and we will write \( E = \frac{\hbar^2 k^2}{2m} \) (or \( E = k^2 \) in the new units) for reasons that will perhaps be clearer in a moment. The Schrödinger operator becomes
\[
L_q = -\frac{d^2}{dx^2} + q(x)
\]
and the differential equation for \( y \) becomes
\[
-\frac{d^2}{dx^2} + q(x)y = k^2 y(x)
\]
Observe that this is a kind of eigenvalue problem for the operator \( L_q \).

When \( q(x) = 0 \) (free particle motion) the solutions take the form
\[
y(x) = A \exp(ikx) + B \exp(-ikx)
\]
If we recall the original ansatz, the solutions of the time-dependent Schrödinger equation corresponding respectively to \( A = 1, B = 0 \) and \( A = 0, B = 1 \) are the functions
\[
\psi_R(x, t) = \exp i(kx - k^2 t)
\]
and
\[
\psi_L(x, t) = \exp(-i(kx + k^2 t))
\]
which describe waves moving respectively to the right and left. If \( q \) is not zero but is, for example, compactly supported, we still expect that for \( x \ll 0 \) or \( x \gg 0 \), \( y(x) \) has the form (1.1).

Indeed, it is natural to expect solutions of the form
\[
y_R(x, k) \sim \begin{cases} e^{ikx} + R_2(k)e^{-ikx} & x \to -\infty \\ T_2(k)e^{ikx} & x \to +\infty \end{cases}
\]
and
\[
y_L(x, k) \sim \begin{cases} e^{-ikx} + R_1(k)e^{ikx} & x \to +\infty \\ T_1(k)e^{-ikx} & x \to -\infty \end{cases}
\]
which represent an incident right-moving (resp. left-moving) wave which has reflected and transmitted components. The coefficients \( R_i(k) \) and \( T_i(k) \) are called transmission and reflection coefficients and are generally complex-valued. Note that, if \( q(x) = 0 \), we have zero reflection and perfect transmission. We will later show that \( T_1(k) = T_2(k) \).

It will turn out that there may also be special solutions which occur for negative energies \( E \), corresponding to purely imaginary \( k \). These solutions are called bound states, and have the asymptotic form
\[
y(x) \sim \begin{cases} e^{-\beta x} & x \to +\infty \\ e^{\beta x} & x \to -\infty \end{cases}
\]
for some \( \beta \). The numbers \( -\beta^2 \) are called the bound state eigenvalues (as \( L_q y = -\beta^2 y \)); we will show later that this may occur for at most finitely many \( \beta \). It follows that the function \( y \) is square-integrable; the number \( c \) for which \( cy \) has norm one
as a vector in $L^2(\mathbb{R})$ is called the norming constant for the bound state. We will also show that $T(k)$ can be determined from either $R_1(k)$ or $R_2(k)$ together with the bound state eigenvalues $-\beta_i^2$ and norming constants $c_i$.

The direct scattering problem consists in studying how the potential $q$ determines the reflection coefficients, bound state eigenvalues, and norming constants. The map

$$q \mapsto \{\beta_i, c_i, R(k)\}$$

is called the direct scattering map. Our first task is to understand and characterize the range of this map for a given class of potential functions $q$. We will see that this map behaves in many respects like a kind of nonlinear Fourier transform of the potential. We will also explore properties of the scattering solutions and prove an important representation formula for the scattering solutions in terms of an integral kernel $B(x, y)$. More precisely, if we factor

$$y_R(x, k) = e^{ikx} m(x, k),$$

then

$$m(x, k) = 1 + \int_0^\infty B(x, y) e^{2iky} dy.$$ 

The remarkable fact is that $B(x, y)$ is independent of $k$. The existence of the representation formula will follow from Hardy space properties of the function $m$.

The inverse scattering problem consists in determining the inverse of the direct scattering map. This will be a more challenging task that culminates in the so-called Gelfand-Levitan Equation, an integral equation which determines the potential $q$ from the data $\{\beta_i, c_i, R(k)\}$. The Gelfand-Levitan-Marchenko equation is an integral equation for the kernel $B(x, y)$ which can be solved for given scattering data. The potential can be recovered directly from the kernel $B(x, y)$ via the formula

2. Direct Scattering

In order to study the scattering problem, we consider the scattering solutions of the eigenvalue equation

$$(2.1) \quad -y''(x) + q(x)y(x) = k^2 y(x).$$

These are functions $f_1(x, k)$ and $f_2(x, k)$ which obey the respective asymptotic conditions

$$(2.2) \quad f_1(x, k) \sim \exp(ikx) \text{ as } x \to +\infty$$

and

$$(2.3) \quad f_2(x, k) \sim \exp(-ikx) \text{ as } k \to -\infty$$

The precise meaning of the condition $y(x) \sim g(x)$ as $x \to \infty$ is that

$$\lim_{x \to \infty} |y(x) - g(x)| = \lim_{x \to \infty} |y'(x) - g'(x)| = 0.$$

We will prove that the Jost solutions $f_1$ and $f_2$ are unique. Comparing with the scattering solutions $y_L$ and $y_R$ discussed heuristically above, we expect that

$$(2.4) \quad f_1(x, k) \sim \frac{1}{T_2(k)} \exp(ikx) + \frac{R_2(k)}{T_2(k)} \exp(-ikx) \text{ as } x \to -\infty$$

and

$$(2.5) \quad f_2(x, k) \sim \frac{1}{T_1(k)} \exp(-ikx) + \frac{R_1(k)}{T_1(k)} \exp(ikx) \text{ as } x \to +\infty.$$
In order to study the Jost solutions it will be helpful to introduce the \( m \)-functions

\[
m_1(x, k) = f_1(x, k) \exp(-ikx) \\
m_2(x, k) = f_2(x, k) \exp(ikx)
\]

These functions solve the singular initial value problems:

\[
m''_1 + 2ikm'_1 = qm_1 \\
m_1(x, k) \sim 1 \text{ as } x \to +\infty
\]

and

\[
m''_2 - 2ikm'_2 = qm_2 \\
m_2(x, k) \sim 1 \text{ as } x \to -\infty
\]

2.1. The Jost Solution. We will solve the problem (2.8) in detail and only summarize the very similar analysis of (2.9). We begin by reformulating (2.8) as an integral equation. To this end, we need the following representation formula. Let

\[
D_k(y) = \int_0^y e^{2ikt} dt
\]

We have the estimates

\[
|D_k(y)| \leq \frac{1}{|k|}
\]

for \( y \leq 0, \text{Im} \, k \geq 0, \text{and } k \neq 0, \text{and}

\[
|D_k(y)| \leq y
\]

for any \( k \) with \( \text{Im} \, k \geq 0. \)

**Lemma 2.1.** Suppose that \( f \) is a measurable real-valued function with the property that \( F(x) = \int_x^\infty |f(y)| \, dy \) is finite for each \( x \). For any \( k \) with \( \text{Im}(k) \geq 0, k \neq 0, \) the unique solution to the singular initial value problem

\[
m''(x) + 2ikm'(x) = f(x) \\
m(x) \sim 1 \text{ as } x \to +\infty
\]

is given by

\[
m(x) = 1 + \int_x^\infty D_k(y-x)f(y) \, dy
\]

If, also, \( G(x) = \int_x^\infty (1 + |y|)|f(y)| \, dy < \infty, \) then the representation formula also holds when \( k = 0. \)

**Proof.** Writing (2.13) as a first-order system we have

\[
\frac{d}{dx} \begin{bmatrix} m(x) \\ m'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2ik \end{bmatrix} \begin{bmatrix} m(x) \\ m'(x) \end{bmatrix} + \begin{bmatrix} 0 \\ f(x) \end{bmatrix}.
\]

A fundamental solution matrix for the homogeneous equation is

\[
Y(x) = \begin{bmatrix} 1 & e^{-2ikx} \\ 0 & -2ike^{-2ikx} \end{bmatrix},
\]
Observe that
\[
Y(x)^{-1} = \begin{bmatrix} 1 & \frac{1}{2ik} \\
0 & \frac{1}{e^{2ikx}} \\
\end{bmatrix}.
\]
Hence, setting
\[
\begin{bmatrix} m(x) \\
m'(x) \\
\end{bmatrix} = Y(x) \begin{bmatrix} n_1(x) \\
n_2(x) \\
\end{bmatrix}
\]
(essentially the “method of variation of parameters”) we conclude that
\[
\frac{d}{dx} \begin{bmatrix} n_1(x) \\
n_2(x) \\
\end{bmatrix} = \frac{1}{2ik} \begin{bmatrix} 1 \\
-1 \\
\end{bmatrix} f(x)
\]
which can be integrated to give
\[
n_1(x) = \frac{1}{2ik} \int_x^\infty f(y) \, dy
\]
and
\[
n_2(x) = -\frac{1}{2ik} \int_x^\infty e^{2iky} f(y) \, dy
\]
(where \(\int_x^\infty\) means an indefinite integral). Thus the general solution takes the form
\[
m(x) = c_1 + c_2 e^{-2ikx} - \frac{1}{2ik} \int_x^\infty \left( e^{2ik(y-x)} - 1 \right) f(y) \, dy
\]
\[
= c_1 + c_2 e^{-2ikx} - \int_x^\infty D_k(y-x) f(y) \, dy
\]
where the convergence of the integral is guaranteed for \(k \neq 0\) by the estimate \((2.11)\) and the finiteness of \(F(x)\), while convergence for \(k = 0\) is guaranteed by the estimate \((2.12)\), and the finiteness of \(G(x)\). □

It follows from Lemma 2.1 that the singular initial value problem \((2.8)\) can be formulated as an integral equation
\[
m(x) = 1 + \int_x^\infty D_k(y-x) q(y) m(y) \, dy
\]
for the Jost solution \(m\). We will first prove existence for \(k \neq 0\), \(\text{Im}(k) \geq 0\), then obtain more detailed estimates and study the behavior of solutions as \(k \to 0\).

First, we will look for continuous solutions assuming that the monotone decreasing function
\[
Q(x) = \int_x^\infty |q(y)| \, dy
\]
is finite for any \(x \in \mathbb{R}\). (we will place more stringent conditions on \(q\) in what follows).

Equation \((2.14)\) takes the form
\[
m = 1 + T m
\]
where
\[
(T\psi)(x) = \int_x^\infty D_k(y-x) q(y) m(y) \, dy.
\]
so formally the solution is given by \( m = (I - T)^{-1} m \). The integral operator \( T \) obeys the bound

\[
| (T\psi)(x) | \leq \frac{Q(x)}{|k|} \sup_{y \geq x} |\psi(y)|
\]

where

\[
Q(x) = \int_{x}^{\infty} |q(y)| \ dy
\]

and hence

\[
| (T^n\psi)(x) | \leq \frac{1}{n!} \frac{Q(x)^n}{n!} \sup_{y \geq x} |\psi(y)|
\]

These estimates show that the Volterra series

\[
m(x, k) = \sum_{n=0}^{\infty} (T^n 1)(x, k).
\]

converges uniformly and absolutely for \( k \neq 0 \) and \( \text{Im}(k) \geq 0 \), so that \( m(x, k) \) is analytic in \( k \) with \( \text{Im}(k) > 0 \) for each fixed \( x \). Thus:

**Theorem 2.2.** For any \( k \neq 0 \) with \( \text{Im}(k) \geq 0 \), the equation \( (2.14) \) has a unique solution with

\[
|m(x, k)| \leq \exp \left( \frac{Q(x)}{|k|} \right)
\]

**Remark 2.3.** It follows easily from \( (2.18) \) that

\[
|m(x, k) - 1| \leq \frac{Q(x)}{|k|} \exp \left( \frac{Q(x)}{|k|} \right)
\]

The estimates on \( m(x, k) \) diverge as \( |k| \to 0 \) but we can refine the estimates to determine the low-energy behavior if we make a stronger assumption on \( q \). Since the general solution for the equation \( m'' = 0 \) (zero energy, no potential) takes the form \( A + Bx \), we should expect any estimates on \( m(x, k) \) that hold uniformly for \( k \) near zero to allow for at least linear growth in the solution as \( x \to -\infty \).

Let

\[
p(x) = (1 + |x|) |q(x)|
\]

and suppose that the monotone decreasing function

\[
P(x) = \int_{x}^{\infty} p(y) \ dy
\]

is finite for each \( x \in \mathbb{R} \).

**Lemma 2.4.** For any \( k \) with \( \text{Im}(k) \geq 0 \), \( k \neq 0 \), the estimate

\[
|m(x, k)| \leq \exp (P(x))
\]

holds for \( x \geq 0 \), while

\[
|m(x, k)| \leq K_2 (1 + |x|) \exp (P(x))
\]

if \( x < 0 \), where

\[
K_2 = 1 + P(0) \exp P(0)
\]
Moreover, the estimate
\[
\left| \frac{\partial m}{\partial x} (x, k) \right| \leq \frac{K_2 P(x) \exp P(x)}{1 + |k|}
\]
holds for any \( k \) with \( \text{Im}(k) \geq 0 \), \( k \neq 0 \), and any \( x \).

**Proof.** We first note that the inequality
\[
|m(x)| \leq 1 + \int_x^\infty (y-x)|q(y)||m(y)| \, dy
\]
holds for any \( x \) and any \( k \) with \( \text{Im}(k) \geq 0 \) by (2.14) and (2.12). First, for \( x \geq 0 \), we have
\[
|m(x)| \leq 1 + \int_x^\infty y |q(y)||m(y)| \, dy
\]
so that by Gronwall’s inequality, for any \( x \geq 0 \),
\[
|m(x)| \leq \exp P(x)
\]
and in particular
\[
|m(x)| \leq \exp P(0)
\]
for any such \( x \). On the other hand, if \( x < 0 \), the same inequality shows that
\[
|m(x)| \leq 1 + \int_x^\infty y |q(y)||m(y)| \, dy
\]
\[
+ |x| \int_x^\infty |q(y)||m(y)| \, dy.
\]
Thus, if \( x \geq -C \),
\[
(2.22) \quad |m(x)| \leq K_2 + C \int_x^\infty |q(y)||m(y)| \, dy
\]
where
\[
K_2 = 1 + P(0) \exp P(0).
\]
Using Gronwall’s inequality again we have
\[
|m(x)| \leq K_2 \exp (CQ(x))
\]
or
\[
|m(x)| \leq K_2 \exp (|x| Q(x))
\]
for any \( x \).

Even if \( Q(x) \) is bounded, this estimate blows up exponentially as \( x \to -\infty \). To improve it, we set
\[
M(x) = \frac{m(x)}{1 + |x|}
\]
and deduce from inequality (2.22) we
\[
|M(x)| \leq K_2 + \int_x^\infty p(y)|M(y)| \, dy
\]
By Gronwall’s inequality yet again
\[
|M(x)| \leq K_2 \exp (P(x)).
\]
Hence, for any \( x < 0 \),
\[
|m(x)| \leq K_2 (1 + |x|) \exp(P(x))
\]
Finally, to estimate the derivative $\partial m/\partial x$, we use the uniform estimate

$$|m(x, k)| \leq K_2 (1 + |x|) \exp(P(x))$$

together with the formula

$$\frac{\partial m}{\partial x}(x, k) = - \int_x^\infty e^{2ik(y-x)} q(y)m(y) \, dy$$

to obtain

$$\left| \frac{\partial m}{\partial x}(x, k) \right| \leq K_2 P(x) \exp(P(x)).$$

For $|k| \geq 1$ we can use the estimates $|m(x, k)| \leq \exp(Q(x))$ and $|D_k(y)| \leq |k|^{-1}$ to conclude that

$$\left| \frac{\partial m}{\partial x}(x, k) \right| \leq \frac{Q(x) \exp(Q(x))}{|k|}.$$

Combining these two estimates we obtain the claimed estimate on $\partial m/\partial x$. □

**Remark 2.5.** We easily conclude from the estimates already proved that

$$|m(x, k) - 1| \leq \frac{1 + |x|}{1 + |k|} K_2 P(x) \exp(P(x))$$

Since $m(x, k)$ is analytic, it is natural to consider the behavior of its derivative. We will obtain an integral equation by differentiating (2.14) with respect to $k$, and so we need estimates on $\partial D_k/\partial k$. From the integral representation (2.10) we have

$$\left| \frac{\partial D_k}{\partial k}(y) \right| = \left| \int_0^y 2ite^{ikt} \, dt \right| \leq y^2$$

if $y < 0$ and $\text{Im}(k) \geq 0$. From the integral equation (2.14) we have

$$(2.23) \quad \frac{\partial m}{\partial k}(x, k) = \int_x^\infty \frac{\partial D_k}{\partial k}(y-x)q(y)m(y) \, dy$$

$$+ \int_x^\infty D_k(y-x)q(y)\frac{\partial m}{\partial k}(y, k) \, dy$$

which is an integral equation for $\partial m/\partial k$ involving the same integral operator $T$ that we have already studied. This equation can also be solved by the Volterra series provided the inhomogeneous term

$$g(x, k) = \int_x^\infty \frac{\partial D_k}{\partial k}(y-x)q(y)m(y) \, dy$$

is bounded. Let

$$r(x) = (1 + |x|)^2 |q(x)|$$

and suppose that the monotone decreasing function

$$R(x) = \int_x^\infty r(y) \, dy$$

is finite for all $x$. Since

$$|g(x, k)| \leq \int_x^\infty (y-x)^2 |q(y)| |m(y)| \, dy,$$

the right-hand side is finite for any $x$. 

Lemma 2.6. For any $k$ with $\text{Im}(k) \geq 0$ and $k \neq 0$, the estimate
\[
\left| \frac{\partial m}{\partial k} (x, k) \right| \leq [R(0) \exp(R(0))] \exp(P(x))
\]
holds for $x \geq 0$, while the estimate
\[
\left| \frac{\partial m}{\partial k} (x, k) \right| \leq \left[ R(0) \exp(R(0)) + 4K_2 |x|^2 P(x) \right] \exp \left( 2P(x) + |x| Q(x) \right)
\]
holds for $x < 0$.

Proof. First let us consider $x \geq 0$. In this case, the inhomogeneous term $g(x, k)$ in (2.23) obeys the estimate
\[
|g(x, k)| \leq \int_0^\infty |y|^2 |q(y)||m(y)| \ dy \leq R(0) \exp(R(0))
\]
Since, for $x \geq 0$,
\[
\left| \frac{\partial m}{\partial k} (x, k) \right| \leq R(0) \exp(R(0)) + \int_x^\infty |q(y)| \left| \frac{\partial m}{\partial k} (y, k) \right| \ dy,
\]
Gronwall’s inequality gives
\[
\left| \frac{\partial m}{\partial k} (x, k) \right| \leq R(0) \exp(R(0)) \exp(P(x))
\]
as claimed.

If $x < 0$ but $x \geq -C$, the inhomogenous term obeys the estimate
\[
|g(x, k)| \leq \int_0^\infty |y|^2 |q(y)||m(y)| \ dy + \int_x^0 (y + C)^2 |q(y)||m(y)| \ dy
\leq K_3(C)
\]
where
\[
K_3(C) = R(0) \exp(R(0)) + 4C^2 P(C) \exp(P(C))K_2
\]
and we have estimated $|m(y)| \leq K_2 (1 + |x|) \exp(P(x))$. Since
\[
\left| \frac{\partial m}{\partial k} (x, k) \right| \leq K_3(C) + \int_x^\infty (y + C)|q(y)| \left| \frac{\partial m}{\partial k} (y, k) \right| \ dy
\]
we can apply Gronwall’s inequality to obtain
\[
\left| \frac{\partial m}{\partial k} (x, k) \right| \leq K_3(C) \exp(P(x) + CQ(x))
\]
for any $x \geq -C$, so
\[
\left| \frac{\partial m}{\partial k} (x, k) \right| \leq \left( K_4 + 4K_2 |x|^2 P(x) \exp(P(x)) \right) \exp(P(x) + |x| Q(x))
\]
where $K_4 = R(0) \exp(R(0))$. This implies the claimed inequality.

We are now ready to investigate the analytic function $k \mapsto m(x, k)$ on the upper half-plane $\text{Im}(k) > 0$. The estimate (2.21) implies that $m(x, k)$ may have at most finitely many zeros since $|m(x, k) - 1| \geq 1/2$ for $|k|$ with
\[
\frac{Q(x)}{|k|} \exp \left( \frac{Q(x)}{|k|} \right) < 1/2.
\]
We will use a spectral argument to show that any such zeros are simple and lie on the imaginary axis.
First, we note that if \( m(a, k) \) has a zero, then the corresponding function \( y(x, k) \) vanishes at \( x = a \) and \( y(x, k) \) decays exponentially as \( x \to +\infty \). The same is true of the derivative \( y' \) (here the prime denotes differentiation with respect to \( x \)). Multiplying the equation
\[-y'' + q(x)y = k^2y\]
by \( \bar{y} \) and integrating by parts we find that
\[
\int_a^\infty \left[ |y'(x)|^2 + q(x)|y(x)|^2 \right] \, dx = k^2 \int_a^\infty |y(x)|^2 \, dx.
\]
This shows that \( k^2 \) must be real. It follows that \( k = i\beta \) for some \( \beta > 0 \). Observe that, if \( k^2 \) is real, the real and imaginary parts of \( y(x) \) are solutions. In case \( y(x) = e^{-\beta x}m(x, i\beta) \), it is not difficult to see that \( \text{Im} \, y(x) \) and \( \text{Im} \, y'(x) \) vanish as \( x \to +\infty \), and hence \( \text{Im} \, y(x) = 0 \), hence \( y(x) \) is real.

Next, we show that any zeros of \( m(x, k) \) with \( \text{Im}(k) > 0 \) are simple. Let \( y(x, k) = \exp(ikx)m(x, k) \) and consider let \( \dot{y}(x, k) = \partial y(x, k)/\partial k \). Then, differentiating the equation \(-y'' + q(x)y = k^2y \) with respect to \( k \) we obtain
\[
(2.24) \quad -\dot{y}'' + q(x)\dot{y} = 2ky + k^2\dot{y}.
\]

Multiplying equation (2.24) by \( \ddot{y} \) and equation (2.24) by \( y \) we see that
\[
\frac{d}{dx} [\dot{y}, y] = 2ky^2
\]
so
\[
2k \int_a^\infty y(x)^2 \, dx = [\dot{y}(x)y'(x) - \dot{y}'(x)y(x)]|_a^\infty
\]
Here we have used the facts that \( y(x) \), \( y'(x) \), \( \dot{y}(x) \), and \( \dot{y}'(x) \) all vanish as \( x \to +\infty \). The vanishing of \( y(x) \), \( y'(x) \), \( \dot{y}(x) \), and \( \dot{y}'(x) \) follow from estimates on \( m(x, k) \), \( m'(x, k) \), and \( \dot{m}(x, k) \). To see that \( \dot{y}'(x) \) vanishes as \( x \to \infty \) we use the estimates on \( m'(x, k) \) and an argument with the Cauchy integral formula. Since \( \dot{y}(a, k) = 0 \) it follows that \( \dot{y}'(a, k) \neq 0 \), since otherwise \( y(x, k) \) would be identically zero by uniqueness. It then follows that \( \dot{y}(a, k) \) is nonzero since the left-hand side is nonzero.

We have proved:

**Lemma 2.7.** For each fixed \( x \), the analytic function \( m(x, k) \) has at most finitely many simple zeros in \( \text{Im}(k) > 0 \) on the imaginary axis.

2.2. **Hardy Space Properties of the Jost Solution.** Next, we investigate Hardy space properties of the functions \( m(x, k) \). We observe that for each fixed \( x \), \( m(x, k) - 1 \) is square-integrable as a function of \( k \), and indeed
\[
\int_{-\infty}^{\infty} |m(x, k) - 1|^2 \, dk \leq (1 + |x|)^2 K_2 P(x)^2 \exp(2P(x))
\]
Moreover, if we let \( k = \zeta + it \) we have
\[
\sup_{t>0} \int_{-\infty}^{\infty} |m(x, \zeta + it) - 1|^2 \, d\zeta \leq (1 + |x|)^2 K_2 P(x)^2 \exp(2P(x))
\]
\footnote{This proof is analogous to the proof of a similar fact for the Dirichlet problem on a finite interval given in Pöschel and Trubowitz, chapter 2, page 30.}
This shows that for each fixed $x$, the function $m(x, \cdot) - 1$ belongs to the Hardy space $H^2_+(\mathbb{R})$. Thus, we have

\begin{equation}
(2.25) \quad m(x, k) - 1 = \int_{0}^{\infty} B(x, y) \exp(ikx) \, dx
\end{equation}

for a function $B(x, y)$ independent of $k$. This kernel will play a fundamental role in the solution of the inverse scattering problem; indeed, the Gelfand-Levitan-Marchenko equation will give a direct method to recover $B(x, y)$, hence $m(x, k)$, and hence $q(x)$, from the scattering data.

For this reason, we will study the properties of $B(x, y)$ in some depth. First, we derive an integral equation for $B(x, y)$ from the fundamental integral equation

\begin{equation}
(2.14)
\end{equation}

We will then use the integral equation to obtain fine estimates on $B(x, y)$. We’ll derive the integral equation formally but prove that the solution of the integral equation coincides with $B$ as defined in (2.25).

Substituting (2.10) in (2.14) we have

\begin{equation}
(2.26) \quad m(x, k) = 1 + \int_{x}^{\infty} \int_{0}^{y-x} e^{2ikz} q(y) m(y, k) \, dz \, dy
\end{equation}

so that by (2.25)

\begin{equation}
(2.27) \quad B(x, y) = \int_{x+y}^{\infty} q(z) \, dz + \int_{0}^{y} \int_{x+y-w}^{\infty} q(z) B(z, w) \, dz \, dw
\end{equation}

Following Deift and Trubowitz [2], we will solve the integral equation for $q$ so chosen that the monotone nonincreasing functions

\begin{equation}
\eta(x) = \int_{x}^{\infty} |q(t)| \, dt
\end{equation}

and

\begin{equation}
\gamma(x) = \int_{x}^{\infty} (t-x) |q(t)| \, dt
\end{equation}

are finite for all $x$. 
Proposition 2.8. The integral equation (2.27) has a unique solution $B$ obeying the estimate

$$|B(x, y)| \leq \eta(x + y)e^{\gamma(x)}$$

The function $B$ is absolutely continuous in $x$ and $y$, and the formulas

$$\frac{\partial B}{\partial x}(x, y) + q(x) = -\int_0^y q(x + y - t)B(x + y - t, t)\, dt$$
$$\frac{\partial B}{\partial y}(x, y) + q(x) = \int_x^\infty q(z)B(z, y)\, dz$$
$$-\int_0^y q(x + y - t)B(x + y - t, t)\, dt$$

hold for almost every $(x, y)$.

Remark 2.9. In particular we have

$$\frac{\partial B}{\partial x}(x, y) - \frac{\partial B}{\partial y}(x, y) = -\int_x^\infty q(z)B(z, y)\, dz$$

which is absolutely continuous even though the first partials need not be absolutely continuous functions.

Proof. We prove convergence of the Volterra-type series $\sum_{n=0}^\infty K^{(n)}(x, y)$ where

$$K^{(0)}(x, y) = \int_{x+y}^\infty q(z)\, dz$$

and

$$K^{(n+1)}(x, y) = \int_0^y \int_{x+y-z}^\infty q(t)B(t, w)\, dt\, dz$$

We will show inductively that

$$|K^{(n)}(x, y)| \leq \frac{\gamma(x)^n}{n!}\eta(x + y)$$

holds, from which convergence together with the estimate (2.28) follow. As $K^{(0)}$ clearly obeys this inequality it suffices to estimate $|K^{(n+1)}(x, y)|$ given the claimed estimate for $K^{(n)}$. From its definition we have

$$|K^{(n+1)}(x, y)| \leq \int_0^y \int_{x+y-z}^\infty |q(t)| |K^{(n)}(t, z)|\, dt\, dz$$
$$\leq \int_0^y \int_{x+y-z}^\infty |q(t)| |\eta(t + z)|\frac{\gamma(t)^n}{n!}\, dt\, dz$$
$$\leq \eta(x + y)\int_0^y \int_{x+y-z}^\infty |q(t)| \frac{\gamma(t)^n}{n!}\, dt\, dz$$

If we write

$$\int_0^y \int_{x+y-z}^\infty (\cdots)\, dt\, dz = \left(\int_0^y \int_{x+y-z}^\infty (\cdots)\, dt\, dz + \int_0^y \int_x^\infty (\cdots)\, dt\, dz\right)$$

then we can estimate the right-hand term in (2.30) as $\eta(x + y)$ times $I_1 + I_2$ where

$$I_1 = \int_0^y \int_{x+y-z}^\infty |q(t)| \frac{\gamma(t)^n}{n!}\, dt\, dz$$
and
\[ I_2 = \int_0^y \int_{x+y}^\infty |q(t)| \frac{\gamma(t)^n}{n!} \, dt \, dz \]

By interchanging orders we have
\[ I_1 = \int_x^{x+y} \int_y^{x+y-t} |q(t)| \frac{\gamma(t)^n}{n!} \, dz \, dt = \int_x^{x+y} (t-x) |q(t)| \frac{\gamma(t)^n}{n!} \, dt \]
while
\[ I_2 \leq \int_{x+y}^\infty (t-x) |q(t)| \frac{\gamma(t)^n}{n!} \, dt. \]

It now follows that
\[ |K^{(n+1)}(x, y)| \leq \int_x^\infty (t-x) |q(t)| \frac{\gamma(t)^n}{n!} \, dt \]
\[ \leq \int_x^\infty (t-x) |q(t)| \left( \int_t^\infty (u-x) |q(u)| \, du \right)^n \]
\[ \leq \int_x^\infty (t-x) |q(t)| \, dt \]

If
\[ F(t) = \int_t^\infty (u-x) |q(u)| \, du \]
then \( F(x) = \gamma(x) \) and
\[ F'(t) = -(t-x) |q(t)| \]
so the right-hand integral in (2.31) is written
\[ -\int_x^\infty F'(t) \frac{F(t)^n}{n!} \, dt = \frac{F(x)^{n+1}}{(n+1)!} \]
We conclude that
\[ |K^{(n+1)}(x, y)| \leq \eta(x+y) \gamma(x)^{n+1} (n+1)! \]
so the estimate (2.29) holds by induction. This estimate shows that the series
\[ \sum_{n=0}^\infty K^{(n)}(x, y) \]
converges to a function \( B(x, y) \) satisfying the estimate (2.28).

It follows that \( B \) is integrable as a function of \( y \) for each \( x \) since
\[ \int_0^\infty |B(x, y)| \, dy \leq e^{\gamma(x)} \int_0^\infty \eta(x+y) \, dy = \gamma(x) e^{\gamma(x)} \]

From the integral equation, it is clear that \( B \) is absolutely continuous in \( x \) and \( y \). The formulas for the first partials \( B_x \) and \( B_y \) follow from differentiating the integral equation (2.27).

\[ \square \]

**Proposition 2.10.** Suppose that \( B \) solves the integral equation (2.27). Then the function \( m \) defined by (2.25) solves the differential equation (2.8).
Proof. We compute
\[ m'(x, k) = \int_0^\infty B_x(x, y)e^{2iky} \, dy \]
\[ = \int_0^\infty [B_x(x, y) - B_y(x, y)] e^{2iky} \, dy + \int_0^\infty B_y(x, y)e^{2iky} \, dy \]
\[ = - \int_0^\infty e^{2iky} \left( \int_x^\infty q(z)B(z, y) \, dz \right) \, dy \]
\[ - B(x, 0) - 2ik \int_0^\infty B(x, y)e^{2iky} \, dy \]
where we have used Remark 2.9 and integration by parts. It follows that \( m'' \) exists for almost every \( x \) and
\[ m''(x, k) + 2ikm(x, k) = -\frac{\partial B}{\partial x}(x, 0) + \int_0^\infty q(x)B(x, y)e^{2iky} \, dy \]
\[ = q(x)m(x, k) \]
for almost every \( x \). □

2.3. Reflection and Transmission Coefficients. From the formula
\[ m_1(x, k) = 1 + \frac{1}{2ik} \int_x^{\infty} \left( e^{2ik(y-x)} - 1 \right) q(y)m_1(y, k) \, dy \]
it follows that for \( x \to -\infty \), we have
\[ m_1(x, k) \sim a(k) + b(k)e^{-2ikx} \]
where
\[ a(k) = 1 - \frac{1}{2ik} \int_{-\infty}^\infty q(y)m_1(y, k) \, dy \]
\[ b(k) = \frac{1}{2ik} \int_{-\infty}^\infty q(y)e^{2iky}m_1(y, k) \, dy. \]
Thus
\[ f_1(x, k) \sim a(k)e^{ikx} + b(k)e^{-ikx} \]
as \( x \to -\infty \), so we may identify
\[ a(k) = \frac{1}{T_2(k)} \]
\[ b(k) = \frac{R_2(k)}{T_2(k)} \]
where \( T_2(k) \) and \( R_2(k) \) are transmission and reflection coefficients. Now let
\[ [u, v] = uv' = u'v \]
(the Wronskian of \( u \) and \( v \)). For \( k \) real, both \( f_1(x, k) \) and \( \overline{f_1(x, k)} \) are solutions. We can use the asymptotics of \( f_1 \) and \( \overline{f_1} \) as \( x \to +\infty \) and \( x \to -\infty \) and the constancy of the Wronskian to compute
\[ -2ik = -2ik \left( |a(k)|^2 - |b(k)|^2 \right) \]
so that
\[ |a(k)|^2 = 1 + |b(k)|^2 \]
We can also compute Proposition 2.11.

For \( k \) real, the functions \( \{f_1(x, k), f_2(x, -k)\} \) form a basis for the two-dimensional space of solutions of \(-y'' + qy = k^2y\), since

\[
[f_1(x, k), f_1(x, -k)] = -2ik.
\]

Similarly, the functions \( \{f_2(x, k), f_2(x, -k)\} \) also form a basis since

\[
[f_2(x, k), f_2(x, -k)] = 2ik
\]

Thus the functions as do the functions \( \{f_2(x, k), f_2(x, -k)\} \), provided that \( k \neq 0 \). Thus, we can write

\[
\begin{align*}
  f_1(x, k) &= \frac{R_2(k)}{T_2(k)} f_2(x, k) + \frac{1}{T_2(k)} f_2(x, -k) \\
  f_2(x, k) &= \frac{R_1(k)}{T_1(k)} f_1(x, k) + \frac{1}{T_1(k)} f_1(x, -k)
\end{align*}
\]

using the asymptotic behaviors of \( f_1 \) and \( f_2 \).

We can obtain a number of relations among these coefficients by exploiting the invariance of the Wronskian. For example, the Wronskian of \( f_1(x, k) \) and \( f_2(x, k) \) is given by

\[
[f_1, f_2] = -2ik \frac{1}{T_2(k)} = -2ik \frac{1}{T_1(k)}
\]

so that \( T_1(k) = T_2(k) \) if \( k \neq 0 \). Henceforth we denote their common value by \( T(k) \).

We can also compute

\[
\begin{align*}
  \frac{R_1(k)}{T_1(k)} &= \frac{1}{2ik} [f_2(x, k), f_1(x, -k)] \\
  \frac{R_2(k)}{T_2(k)} &= \frac{1}{2ik} [f_2(x, -k), f_1(x, k)]
\end{align*}
\]

By uniqueness we also have the relations

\[
\begin{align*}
  f_1(x, -k) &= \frac{T_1(x, k)}{T_1(k)} \\
  f_2(x, -k) &= \frac{T_2(x, k)}{T_2(k)}
\end{align*}
\]

for \( k \) real.

From these relations we easily deduce:

**Proposition 2.11.** For \( k \) real, the following identities hold:

(i): \( T_1(k) = T_2(k) \). We denote their common value by \( T(k) \).

(ii): \( R_1(k) T_2(-k) + R_2(k) T_1(-k) = 0 \)

(iii): \( T(k) = T(-k), R_j(k) = R_j(-k) \) for \( j = 1, 2 \).

(iv): \( |T(k)|^2 + |R_1(k)|^2 = |T(k)|^2 + |R_2(k)|^2 = 1 \)

The matrix

\[
S(k) = \begin{bmatrix}
  T_1(k) & R_2(k) \\
  R_1(k) & T_2(k)
\end{bmatrix}
\]

is called the *scattering matrix* for the one-dimensional Schrödinger equation. It follows from the Proposition that \( S(k) \) is a unitary matrix for each \( k \).
From the representation formulas above we have the asymptotics

$$\frac{1}{T(k)} = 1 - \frac{1}{2ik} \int q(y) + O\left(\frac{1}{k^2}\right)$$

$$\frac{R_2(k)}{T(k)} = \frac{1}{2ik} \int e^{2iky} q(y) \, dy + O\left(\frac{1}{k^2}\right)$$

as $|k| \to \infty$.

The representation formula (2.32) shows that the function $T(k)^{-1}$ is analytic in $\text{Im}(k) > 0$ provided that $q(x)$ has sufficiently rapid decrease as $|x| \to \infty$ since $m_1(x, k)$ is analytic in $k$ (the decay condition on $x$ is needed to ensure that $m(x, k)$ is bounded or at most polynomially growing so that the integral for $T(k)^{-1}$ converges). It is natural to ask whether this analytic function may have zeros (corresponding to poles of $T(k)$). From the formula (2.34) it is clear that $T(k)$ has a pole if and only if $[f_1, f_2] = 0$, i.e., $f_1(x, z_0) = cf_2(x, z_0)$ for some $z_0$ with $\text{Im}(z_0) > 0$. Note that $f_1$ is exponentially decaying as $x \to +\infty$ and $f_2$ is exponentially decaying as $x \to -\infty$ so that $f_1(x, z_0)$ decays exponentially and is hence square-integrable. This implies that $z_0^2$ must be real owing to the following lemma.

**Lemma 2.12.** Suppose that $q$ is real and that $y$ is a square-integrable solution to the equation $-y'' + qy = \lambda y$. Then $\lambda$ is real.

**Proof.** Multiply by $\overline{y}$ and integrate by parts to obtain the formula

$$\int_{-\infty}^{\infty} |y'(x)|^2 \, dx + \int_{-\infty}^{\infty} q(x) |y(x)|^2 \, dx = \lambda \int_{-\infty}^{\infty} |y(x)|^2 \, dx$$

from which reality of $\lambda$ follows. \qed

We also claim that zeros of $[f_1, f_2]$ are simple. It suffices so show that if $[f_1, f_2](z_0) = 0$ then its derivative at $z_0$ is nonzero. For any family of solutions of $-y'' + qy = k^2 y$ smooth in $k$ we have

$$-y'' + qy = k^2 y + 2ky$$

so

$$\frac{d}{dx} [f_1, f_2] = -2kf_1f_2$$

and

$$\frac{d}{dx} [\dot{f}_1, f_2] = 2kf_1f_2$$

Using the facts that $[f_1, f_2]$ is independent of $x$ and that $f_1$ and $f_2$ vanish as $x \to \pm \infty$ if $k = z_0$ we can integrate (2.37) from $-\infty$ to $0$ and (2.38) from $0$ to $\infty$ to obtain

$$\frac{d}{dz} [f_1, f_2](z_0) = \left[ f_1, \dot{f}_2 \right](z_0) + \left[ \dot{f}_1, f_2 \right](z_0)$$

$$= 2z_0 \int_{-\infty}^{\infty} f_1(x, z_0) f_2(x, z_0) \, dx$$

This integral is nonzero since $f_1(x, z_0)$ is a multiple of $f_2(x, z_0)$.

From the formula

$$[f_1, f_2] = \frac{-2ik}{T(k)}$$
we see that $T(k)$ has a simple pole at $z_0$ with residue
\begin{equation}
(2.39)\quad -i \left( \int_{-\infty}^{\infty} f_1(x, z_0)f_2(x, z_0) \, dx \right)^{-1}
\end{equation}

From the asymptotics (2.35) it is clear that $T(k)$ has no zeros for $|k|$ sufficiently large.

It remains to study the behavior of $T(k)$ as $|k| \to 0$. From the representation formula (2.32) there are two cases: either $\int q(x)m_1(x, 0) \, dx \neq 0$ and $T(k)$ vanishes as $|k| \to 0$, or $\int q(x)m_1(x, 0) \, dx = 0$ and $T(k)$ has a nonzero limit as $|k| \to 0$. Note that, in all cases, $|T(k)| \leq 1$.

Let $I_1(k) = \int_{-\infty}^{\infty} q(x)m_1(x, k) \, dx$, a function analytic in $\text{Im}(k) > 0$ and continuous for $\text{Im} k \geq 0$. It follows from the estimates on $\dot{m}(x, k)$ that $I_1(k)$ is also differentiable at $k = 0$. If $I_1(0) \neq 0$ we have
\begin{equation}
(2.40)\quad T(k) = \frac{2ik}{2ik - I_1(0) + O(k)} = \alpha k + o(k)
\end{equation}
where $\alpha = -2iI_1(0)^{-1}$. If $I_1(0) = 0$ we may write
\begin{equation}
(2.41)\quad T(k) = \frac{2ik}{2ik - I'_1(0)k + O(k^2)}
\end{equation}
where $I'_1(0) \neq 2i$ since $|T(k)| \leq 1$ for $k$ real. It follows that
\begin{equation}
T(0) = 2i/(2i - I'_1(0)).
\end{equation}

Since either $T(0)$ or $T'(0)$ is nonvanishing, it follows that zeros of $T$ cannot accumulate at $k = 0$. Putting this all together, we conclude:

**Proposition 2.13.** The function $T(k)$ is meromorphic in $\text{Im}(k) > 0$ and has at most finitely many simple poles in $\text{Im}(k) > 0$ with residues given by (2.39).

The reflection coefficient does not generally extend to an analytic function on the upper half-plane. (One exception is the case where $q$ has compact support.) Let $I_2(k) = \int_{-\infty}^{\infty} q(y)e^{2iky}m_1(y, k) \, dy$. From the representation formula
\begin{equation}
(2.42)\quad \frac{R_2(k)}{T(k)} = \frac{1}{2ik} \int_{-\infty}^{\infty} q(y)e^{2iky}m_1(y, k) \, dy
\end{equation}
we see that there are two possibilities for the behavior of $R(k)$ as $k \to 0$.

In the first case, $I_2(0)$ and hence $I_1(0)$ are nonzero, so that $T(k)$ has a first-order zero at $k = 0$ and by (2.40) and (2.42)
\begin{equation}
\lim_{k \to 0} R_2(k) = -1
\end{equation}

In the second case, $I_2(0) = I_1(0) = 0$ and using (2.41) and (2.42) together with the formula $I'_2(0) = I'_1(0)$ (for $q$ with sufficiently rapid decay at infinity) we conclude
\begin{equation}
\lim_{k \to 0} R_2(k) = \frac{1}{2i - I'_1(0)} I'_1(0)
\end{equation}

Finally we discuss some further regularity properties of the reflection and transmission coefficients. It will be useful to note that, from the representation for $m_1$
in terms of $B_1$, we have
\[
\frac{R_2(k)}{T(k)} = \frac{1}{2ik} \int_{-\infty}^{\infty} q(y) e^{2iky} \, dy + \frac{1}{2ik} \int_{-\infty}^{\infty} \int_{\infty}^{\infty} q(y) e^{2ikz} B_1(x, z - y) \, dz \, dy
\]
\[
= \frac{1}{2ik} \int_{-\infty}^{\infty} e^{2iky} \left[ q(y) + \int_{z}^{\infty} B_1(x, z - y) \, dz \right] \, dy
\]

2.4. Scattering Data. In this subsection we will prove that one reflection coefficient together with the eigenvalues and norming constants suffice to determine the transmission coefficient and the other reflection coefficient. We will first consider the case where there are no eigenvalues so that $T(k)$ is a holomorphic function in $\Im k > 0$. In this case, $T(k) - 1$ is holomorphic and, for $k$ real, and $T(k) - 1 = \mathcal{O}(|k|^{-1})$ as $|k| \to \infty$ for $k$ real owing to the asymptotic estimate (2.35). It follows that $T(k) - 1 \in H^2_+(\mathbb{R})$. Using Hardy space properties of $T(k)$ we will prove:

**Proposition 2.14.** Suppose that $q$ has no bound states. Then the scattering matrix can be reconstructed from either $R_1(k)$ or $R_2(k)$.

**Proof.** We will show how to recover $T(k)$ from the values of $|T(k)|$ on the real axis, which are known from $R_1(k)$ or $R_2(k)$ through the unitarity relation in Proposition 2.11(iv). Given either of $R_1$ or $R_2$ and $T$, we can reconstruct the other reflection coefficient by solving unitarity relation in Proposition 2.11(ii). Before we give details of the proof we outline the main ideas.

The idea will be to use the boundary values of $|T(k)|$ on the real axis to construct a function, analytic in $\Im(k) > 0$, with the same boundary values as $T$, and then use analyticity to show that these functions must coincide.

Let
\[
h(k) = \exp \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\log |T(\zeta)|}{\zeta - k} \, d\zeta \right)
\]
The integrand is integrable for $\Im(k) > 0$ since $\log |T(\zeta)|$ is locally integrable and $\log |T(\zeta)| = \mathcal{O} \left( |k|^{-1} \right)$ as $|k| \to \infty$. Thus $h(k)$ is analytic in $\Im(k) > 0$ and
\[
\log |h(k)| = \frac{b}{\pi} \int_{-\infty}^{\infty} \frac{\log |T(\zeta)|}{(\zeta - a)^2 + b^2} \, d\zeta
\]
The lemma below will show that $\log |T(k)|$ is given by the same expression, so that $|h(k)| = |T(k)|$ and the ratio $h(k)/T(k)$ is an entire function of constant modulus. It follows from the maximum modulus principle that $h(k)/T(k)$ is constant. Using the asymptotics of each function as $|k| \to \infty$ we conclude that the ratio is one so that
\[
T(k) = \exp \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\log |T(\zeta)|}{\zeta - k} \, d\zeta \right)
\]
which shows that $T(k)$ can be recovered either from $R_1(k)$ or from $R_2(k)$. \hfill \Box

**Remark 2.15.** Although we don’t prove it here, it can be shown that if $T(k)$ has known simple poles, these together with the reflection coefficient suffice to reconstruct $T(k)$.

Next, we prove the lemma from complex analysis that is used in the proof above. The lemma and its proof come from [2].
Lemma 2.16. Suppose that $h(k)$ is analytic in $\text{Im}(k) > 0$, and

(i) $|h(k)| \leq 1$,
(ii) $h(k)$ has no zeros in $\text{Im}(k) \geq 0$, $k \neq 0$,
(iii) $|h(k)| \geq C|k|$ for $k$ in a neighborhood of $k = 0$, and
(iv) $h(k) = 1 + O\left(|k|^{-1}\right)$ as $|k| \to \infty$ uniformly in $k$ with $\text{Im}(k) \geq 0$.

Then the formula

$$
\log |h(k)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |h(\zeta)|}{(\zeta - x)^2 + y^2} \, d\zeta
$$

holds for all $k = x + iy$ with $\text{Im}(k) > 0$.

Proof. Let $h_\beta(k) = h(k + i\beta)$. Then $h_\beta$ is analytic in $\text{Im}(k) \geq 0$ and $c < |h_\beta(k)| < 1$ for $c > 0$. It follows that $\log h_\beta(k)$ is defined and analytic in $\text{Im}(k) \geq 0$ since we can integrate the analytic function $h_\beta'(z)/h_\beta(z)$ and, for sufficiently large $|z|$ we have $|h(z) - 1| < 1/2$ say. The function $\log h_\beta(z)$ is analytic in $\text{Im}(z) > 0$, so it follows from Cauchy’s formula and the asymptotics of $\log h_\beta(z)$ as $|z| \to \infty$ that:

$$
\log h_\beta(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log h_\beta(\zeta)}{\zeta - k} \, d\zeta
$$

while

$$
0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log h_\beta(\zeta)}{\zeta - k} \, d\zeta.
$$

We conclude that

$$
\log h_\beta(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log h_\beta(\zeta) + \log h_\beta(\zeta)}{\zeta - k} \, d\zeta
$$

$$
= \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\log |h_\beta(\zeta)|}{\zeta - z} \, d\zeta.
$$

Hence, taking real parts on both sides and setting $k = x + iy$,

$$
\log |h_\beta(k)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |h_\beta(\zeta)|}{(\zeta - x)^2 + y^2} \, d\zeta
$$

To prove the representation formula we wish to take limits as $\beta \downarrow 0$. The limit of the left-hand side is $\log h(k)$ by the analyticity of $\log h(z)$; it remains to justify the interchange of limit on the right-hand side. For $|\zeta| > \delta$ we have $c \leq |h_\beta(\zeta)| \leq 1$ for some constant $c > 0$ uniformly in $\beta > 0$ (since $h$ has no zeros away from zero and $h(\zeta) \to 1$ as $|\zeta| \to \infty$) so $\log |h_\beta(\zeta)| = O\left(|\zeta|^{-1}\right)$ uniformly in $\beta > 0$. It follows from the Dominated Convergence Theorem that

$$
\lim_{\beta \downarrow 0} \frac{1}{2\pi i} \int_{|\zeta| > \delta} \frac{\log |h_\beta(\zeta)|}{(\zeta - x)^2 + y^2} \, d\zeta = \frac{1}{2\pi i} \int_{|\zeta| > \delta} \frac{\log |h(\zeta)|}{(\zeta - x)^2 + y^2} \, d\zeta
$$

On the other hand, if $|\zeta| < \delta$, we have $\log |h_\beta(\zeta)| \geq \log C - \log |\zeta|$ for all sufficiently small $\beta > 0$, so we can use the Dominated Convergence Theorem again to conclude that

$$
\lim_{\beta \downarrow 0} \frac{1}{2\pi i} \int_{|\zeta| < \delta} \frac{\log |h_\beta(\zeta)|}{(\zeta - x)^2 + y^2} \, d\zeta = \frac{1}{2\pi i} \int_{|\zeta| < \delta} \frac{\log |h(\zeta)|}{(\zeta - x)^2 + y^2} \, d\zeta
$$

These two results together give the representation formula as claimed. □
3. The Inverse Problem

[Warning: This section is under construction]

In this section we will study the inverse problem in the absence of bound states, i.e., when \( T(k) \) is holomorphic in the upper half-plane. The problem is to study invertibility of the map

\[
q(x) \mapsto r(k)
\]

for \( q \) belonging to a specified function space. We will only consider the case where \( q \) has no bound states and \( q \) lies in the space \( S(\mathbb{R}) \). Note that, here and in what follows \( r(k) \) denotes one of the two possible reflection coefficients which we choose and fix. This problem consists of several parts:

1. Describe the range of the direct map, i.e., give necessary and sufficient conditions for a function \( r(k) \) to be the reflection coefficient of a potential \( q \in S(\mathbb{R}) \) without bound states.

2. Show that the map is one-to-one, i.e., show that if \( q_1 \) and \( q_2 \) are potentials in this class with reflection coefficients \( r_1(k) \) and \( r_2(k) \) and \( r_1(k) = r_2(k) \), then \( q_1 = q_2 \) a.e.

3. Give a procedure for finding \( q(x) \) given the reflection coefficient \( R(k) \).

We will discuss the second and third parts of this problem and return to the first part once we have advanced tools in hand to analyze the range of the direct scattering map. As always, complex analysis plays a fundamental role. We begin with the following “vanishing lemma” from [2]; the proof we give is also taken from this paper.

**Lemma 3.1.** Suppose that \( h(k) \in H^2_+(\mathbb{R}) \) and that, for some measurable function \( r(k) \) with \(|r(k)| < 1 \) a.e., \( h(k) + r(k)h(k) \in H^2_+(\mathbb{R}) \). Then \( h(k) = 0 \).

**Proof.** We will show that

\[
\int_{-\infty}^{\infty} (1 - |r(k)|) |h(k)|^2 \, dk = 0
\]

from which it follows that \( h(k) = 0 \) a.e. Estimate

\[
\int_{-\infty}^{\infty} (1 - |r(k)|) |h(k)|^2 \, dk \leq \int_{-\infty}^{\infty} |h(k)|^2 \, dk - \left| \int_{-\infty}^{\infty} r(k)h^2(k) \, dk \right|
\]

\[
\leq \left| \int_{-\infty}^{\infty} |h(k)|^2 + r(k)h^2(k) \, dk \right|
\]

\[
= \left| \int_{-\infty}^{\infty} h(k) \left( r(k)h(k) + \overline{h(k)} \right) \, dk \right|
\]

\[
= 2\pi \int_{-\infty}^{\infty} \hat{h}(k)(rh(h) + \overline{h})(-k) \, dk
\]

\[
= 0
\]

The last step follows since both \( \hat{h} \) and \( rh + \overline{h} \) have support on the same half-line. \( \square \)

As a first application, we prove, following Deift and Trubowitz [2]:

**Theorem 3.2.** Suppose that \( q_1 \) and \( q_2 \) are potentials in \( L^{1.2} \) without bound states and \( r_1 = r_2 \). Then \( q_1 = q_2 \).
Proof. Consider the functions $h_i(k) = m_i(x; k; q_1) - m_i(x, k, q_2)$. Then $h_i \in H^2_+(\mathbb{R})$ for each fixed $x$ as functions of $k$. On the other hand, setting $r_1 = r_2 = r$, and using the fact that $t_1 = t_2 = t$, we have $r(k)e^{2ikx}h_1(k) + \overline{r_2(k)} = t(k)h_2(k) \in H^2_+ (\mathbb{R})$ since $t(k)$ is bounded and analytic in $\text{Im}(k) > 0$ and $h_2 \in H^2_+ (\mathbb{R})$. It follows from the lemma above that $h_1 = 0$. Hence $m_1(x, k; q_1) = m_2(x, k, q_2)$. As $q = m''/m + 2ikm'/m$ we conclude that $q_1 = q_2$. \[\Box\]

3.1. The Gelfand-Levitan-Marchenko Equation. From the relation

$$f_2(x, k) = \frac{R_1(k)}{T(k)} f_1(x, k) + \frac{1}{T(k)} f_1(x, -k)$$

it follows that

$$(3.1) \quad T(k)m_2(x, k) = R_1(k)e^{2ikx}m_1(x, k) + m_1(x, -k)$$

Since

$$T(k)m_2(x, k) = [T(k) - 1]m_2(x, k) + [m_2(x, k) - 1] + 1$$

it follows that $T(k)m_2(x, k) - 1 \in H^2_+(\mathbb{R})$ and hence, also that

$$(3.2) \quad [m_1(x, -k) - 1] + R_1(k)e^{2ikx}m_1(x, k) \in H^2_+(\mathbb{R})$$

by (3.1). Now define

$$F_1(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{2ikx} R_1(k) \, dk$$

so

$$R_1(k) = \int_{-\infty}^{\infty} e^{-2ikx} F_1(x) \, dx$$

and recall that

$$m_1(x, k) = 1 + \int_{0}^{\infty} B_1(x, y)e^{2iky} \, dy.$$ 

Note that, by the reality condition $R_1(k) = R_1(-k)$, the function $F_1(k)$ is real. Then the left-hand side of (3.2) is given by

$$\int_{0}^{\infty} B_1(x, y)e^{-2iky} \, dy + \int_{-\infty}^{0} \int_{0}^{\infty} e^{2ik(y-t-x)} F_1(t)B(x, y) \, dy \, dt + \int_{-\infty}^{\infty} e^{-2ik(t-x)} F_1(t) \, dt$$

or

$$\int_{0}^{\infty} B_1(x, y)e^{-2iky} \, dy + \int_{-\infty}^{0} \int_{0}^{\infty} e^{-2iky} F_1(x+s+y)B(x, s) \, ds \, dy + \int_{-\infty}^{\infty} e^{-2iky} F_1(x+y) \, dy$$

In order that the sum of these integrals define an $H^2_+(\mathbb{R})$ function, the integrands must vanish for $y > 0$. We conclude that

$$(3.3) \quad F_1(x+y) + B_1(x, y) + \int_{0}^{\infty} F_1(x+y+s)B(x, s) \, ds = 0$$

for $y > 0$.

Equation (3.3) is the Gelfand-Levitan-Marchenko equation. It is an equation which, for each fixed $x$, is to be solved in $L^1(0, \infty)$ for the unknown function $B_1(x, y)$. If we fix $x$ and set

$$F_1(x+y) = f(y), \quad B(x, y) = b(y)$$

we have $f + b + Tb = 0$ where

$$(T\psi)(y) = \int_{0}^{\infty} f(y+t)\psi(t) \, dt.$$
The formal solution is given by \( b = -(I + T)^{-1}f \) but we need to show that the inverse exists and study its mapping properties. These depend on properties of \( f \) (or equivalently \( F_1 \)).

**Lemma 3.3.** For any \( q \in S(\mathbb{R}) \), \( R(k) \in S(\mathbb{R}) \).

**Proof.** We will show that \( R(k)/T(k) \) and \( T(k) - 1 \) belong to \( S(\mathbb{R}) \) from which the conclusion follows. First, we note that if \( q \in S(\mathbb{R}) \), it is not difficult to see from the integral equation for \( m \) that \( m(x, k) \) is \( C^\infty \) in \( x \) and \( k \) with bounded derivatives of all orders, and rapidly decaying in \( k \) as \( |k| \to \infty \). We can then use the representation formulas (2.32) and (2.33) for \( a(k) \) and \( b(k) \) (alias \( 1/T(k) \) and \( R(k)/T(k) \)) to conclude that these functions also lie in \( S(\mathbb{R}) \). \( \square \)

Since the Fourier transform takes \( S(\mathbb{R}) \) into itself, it follows that, also \( F_1 \in S(\mathbb{R}) \).

In particular, \( T : L^1(0, \infty) \to L^1(0, \infty) \) with \( \|T\| \leq \|F_1\|_1 \). Moreover:

**Lemma 3.4.** The operator \( T \) is a compact operator from \( L^1(0, \infty) \) to itself.

**Proof.** We will use Proposition E.5 from Appendix E. It suffices to show that the unit ball \( S = \{ \psi \in L^1(0, \infty) : \|\psi\|_1 \leq 1 \} \) is mapped to a compact set by \( T \). For any \( \psi \in S \) we have

\[
(T\psi)(t + h) - (T\psi)(t) = \int [f(y + t + h) - f(y + t)] \psi(t) \, dt
\]

so

\[
\|(T\psi)(\cdot + h) - (T\psi)(\cdot)\|_1 \leq \|f(\cdot + h) - f(\cdot)\|_1 \|\psi\|_1
\]

which goes to zero as \( h \to 0 \) uniformly in \( \psi \in S \) by the continuity of translations in \( L^1 \). Next, compute

\[
\int_R \|(T\psi)(y)\| \, dy \leq \int_0^\infty \left( \int_R \|f(t + y)\| \, dy \right) |\psi(t)| \, dt \\
\leq \left( \int_R \|f(y)\| \, dy \right) \|\psi\|_1
\]

which again goes to zero as \( R \to \infty \) uniformly in \( \psi \in S \). It follows that \( T \) is compact. \( \square \)

Next, we show:

**Lemma 3.5.** Suppose that \( f \in L^1 \cap L^2 \) and \( (I + T)\psi = 0 \) for \( \psi \in L^1(0, \infty) \). Then \( \psi = 0 \) a.e.

**Proof.** First, since \( F_1 \) is real, we may assume that \( \psi \) is real since \( T \) commutes with complex conjugation. If \( \psi \in L^1(0, \infty) \) and \( f \in L^1 \cap L^2 \) it follows from Minkowski’s integral inequality that \( \psi \in L^2(0, \infty) \) also. Let

\[
h(k) = \int_0^\infty e^{2ikt} \psi(t) \, dt
\]

so that \( h \in H_2^2(\mathbb{R}) \),

\[
\psi(y) = \frac{1}{\pi} \int e^{-2iky} h(k) \, dk
\]

and \( \overline{h(k)} = h(-k) \). Observe that \( h(k) + r(k)h(k) \) is an \( L^2 \)-function since \( h \in L^2 \) and \( |r(k)| < 1 \). Substituting in the equation \( \psi + T\psi = 0 \) we get that, for any \( y > 0 \),

\[
\int_{-\infty}^\infty e^{2iky} \left[ \overline{h(k)} + r(k)h(k) \right] \, dk = 0
\]
It follows that for some function \( \varphi \in L^2(-\infty, 0) \), the identity
\[
\overline{h(k)} + r(k)h(k) = \frac{1}{\pi} \int_{-\infty}^{0} e^{-2iky} \varphi(y) \, dy
\]
holds. Using this representation it is easy to see that \( \overline{h(k)} \, dk + r(k)h(k) \in H^2_+(\mathbb{R}) \).

We now apply Lemma 3.1 to conclude that \( h = 0 \) a.e. \( \square \)

From the Fredholm alternative (Theorem B.4), we conclude that \( (I + T) \) is invertible, and that the equation \( f + b + Tb = 0 \) has a unique solution \( b \in L^1(0, \infty) \).

From this discussion we conclude that the Gelfand-Levitan-Marchenko equation has a unique solution \( B(x, y) \) with \( B(x, y) \in L^1(0, \infty) \) for each fixed \( x \). In fact, in our case, much more is true since \( F_1 \in \mathcal{S}(\mathbb{R}) \): we have that \( B(x, y) \)

\section*{Appendix A. Notation Index}

- \( D_k \): Fundamental solution kernel for the operator \( d^2/dx^2 + 2ik(d/dx) \); see (2.10).
- \( f_1 \): Scattering solution of the Schrödinger equation (2.1) with \( f_1(x, k) \sim e^{ikx} \) as \( x \to +\infty \); see (2.2).
- \( f_2 \): Scattering solution of the Schrödinger equation (2.1) with \( f_2(x, k) \sim e^{-ikx} \) as \( x \to -\infty \); see (2.3).
- \( m \): Jost solution (denoted \( m_1 \) in (2.8); see (2.14).
- \( m_1 \): Jost solution with boundary condition at \( +\infty \); see (2.6).
- \( m_2 \): Jost solution with boundary condition at \( -\infty \); see (2.7).
- \( T \): Integral operator in fundamental integral equation (2.14); see (2.15).

\section*{Appendix B. Gronwall’s Inequality}

\textbf{Theorem B.1.} Suppose that \( u : [a, b] \to \mathbb{R}^+ \) and \( B : [a, b] \to \mathbb{R}^+ \) are continuous, that \( A \geq 0 \), and that the inequality
\[
u(t) \leq A + \int_{a}^{t} B(s)u(s) \, ds
\]
holds on \([a, b]\). Then
\[
u(t) \leq A \exp \left( \int_{a}^{t} B(s) \, ds \right)
\]

\textbf{Proof.} Suppose first that \( A > 0 \) strictly and consider the strictly positive function
\[
F(t) = A + \int_{a}^{t} B(s)u(s) \, ds.
\]
This function is continuous and differentiable on \([a, b]\) with
\[
F'(t) = B(t)u(t) \leq B(t)F(t)
\]
so that
\[
\frac{d}{dt} \log F(t) \leq B(t)
\]
from which it follows that
\[
\log F(t) \leq \log F(0) + \int_{0}^{t} B(s) \, ds.
\]

\textsuperscript{3}This proof is taken from Terence Tao’s CBMS lectures, chapter 1.2.
If $A = 0$ then for any $\varepsilon > 0$, $u(t) \leq \varepsilon \exp \left( \int_a^b B(s) \, ds \right)$ so $u(t) = 0$ on $[a, b]$.

**Appendix C. The Fourier Transform**

The following is a brief summary, without proofs, of the main facts about the Fourier transform which are used in the notes. Recall that $S(\mathbb{R})$ denotes the Schwartz space of smooth functions of rapid decrease. For $f \in S(\mathbb{R})$ we define

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x) \, dx$$

and for $g \in S(\mathbb{R})$ we define

$$\check{g}(x) = (\mathcal{F}^{-1}g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} g(\xi) \, d\xi$$

The mapping $\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$ is an isomorphism with inverse $\mathcal{F}^{-1}$. Moreover,

$$\|\mathcal{F}f\|_{L^2(\mathbb{R})}^2 = 2\pi \|f\|_{L^2(\mathbb{R})}^2$$

This equality can be used to extend the Fourier transform to a bounded linear mapping from $L^2$ to itself. For general $f \in L^2$ we have

$$\hat{f}(\xi) = \text{l.i.m.}_{R \to \infty} \left( \int_{-R}^{R} e^{-i\xi x} f(x) \, dx \right)$$

where l.i.m. denotes “limit in the mean”: calling the function in parentheses $g_R$ one means that $\|\hat{f} - g_R\|_2 \to 0$ as $R \to \infty$, where $\| \cdot \|_2$ denotes the $L^2$ norm.

Recall that the convolution of two functions $f$ and $g$ is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t) \, dt$$

It is well-defined for $f$ and $g$ belonging to $L^1$. Using Minkowski’s integral inequality one can show that if $f \in L^1$ and $g \in L^p$, $1 \leq p \leq \infty$, then $f * g$ belongs to $L^p$ with $\|f * g\| \leq \|f\|_1 \|g\|_p$.

Then

$$\hat{f} * \check{g}(\xi) = \hat{f}(\xi)\check{g}(\xi)$$

Conversely,

$$\mathcal{F}^{-1}(gh)(x) = (\check{g} * \check{h})(x).$$

**Appendix D. Hardy Spaces**

Suppose that $h$ is analytic in the upper half-plane $\operatorname{Im}(z) > 0$. We say that $h \in H^2_+(\mathbb{R})$ if

$$\|h\|_{H^2_+}^2 = \sup_{b > 0} \int_{-\infty}^{\infty} |h(s + it)|^2 \, ds$$

is finite.

To see that this space is nonempty, let $f \in L^2(\mathbb{R})$ and set

$$h(z) = \frac{1}{2\pi i} \int \frac{1}{z - \zeta} f(\zeta) \, d\zeta$$
for $\text{Im}(z) > 0$. This integral converges, by Hölder’s inequality since $f \in L^2$ and $(z - \zeta)^{-1}$ also belongs to $L^2$ as a function of $\zeta$, and indeed

$$h(z) = \lim_{R \to \infty} g_R(z)$$

where

$$g_R(z) = \frac{1}{2\pi i} \int_{-R}^{R} \frac{1}{z - \zeta} f(\zeta) \, d\zeta$$

and the limit is uniform in $z$ with $\text{Im}(z) > c_0 > 0$. It is not difficult to see that the functions $g_R$ are analytic so $h$, as a uniform limit of analytic functions, is analytic.

Moreover,

$$h_0(x) = \lim_{\varepsilon \to 0} h(x + i\varepsilon)$$

exists in $L^2$-sense. To compute the limit it is helpful to consider the Fourier transform of $h$:

$$\hat{h}_b(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} h(x + ib) \, dx$$

Since $h$ is a convolution and the formula

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{t + ib} e^{-it\xi} \, dt = \begin{cases} e^{-b\xi} & \xi > 0 \\ 0 & \xi < 0 \end{cases}$$

holds for $z = a + ib$, it follows from the convolution theorem that

$$\hat{h}_b(\xi) = \begin{cases} e^{-b\xi} \hat{f}(\xi) & \xi > 0 \\ 0 & \xi < 0 \end{cases}$$

Thus $h_0 := \lim_{b \to 0} h_b$ exists in $L^2$ by the Dominated Convergence Theorem, and

$$\hat{h}_0(\xi) = \begin{cases} \hat{f}(\xi) & \xi > 0 \\ 0 & \xi < 0 \end{cases}$$

and

$$\|h\|_{H^2_+} = \|h_0\|_2$$

The following theorem shows that, in fact, any $h \in H^2_+(\mathbb{R})$ arises in this way, and if $h_0$ denotes the $L^2$-boundary value of $h$, the correspondence $h \leftrightarrow h_0$ is an isometry, and we can think of $H^2_+(\mathbb{R})$ as the set of functions $f \in L^2(\mathbb{R})$ with $\hat{f}(\xi) = 0$ for $\xi < 0$.

**Theorem D.1.** Suppose that $h \in H^2_+(\mathbb{R})$. There is a unique $f \in L^2(\mathbb{R})$ with $\hat{f}(\xi) = 0$ for a.e. $\xi < 0$ with the property that

$$h(z) = \frac{1}{2\pi i} \int \frac{1}{z - \zeta} f(\zeta) \, d\zeta$$

for all $z$ with $\text{Im}(z) > 0$, and the equality

$$\|h\|_{H^2_+} = \|f\|_{L^2(\mathbb{R})}$$

holds.
Proof. Suppose that \( \hat{h} \in H^2_b(\mathbb{R}) \), let \( h_b \) denote its restriction to the line \( \text{Im}(z) = b \).
In order to recover the boundary value \( f \), we will show that \( e^{ib\xi} \hat{h}_b(\xi) \) is independent of \( b \) and define \( f \) by the equation \( \hat{f}(\xi) = e^{ib\xi} \hat{h}_b(\xi) \). We will then show that \( f \in L^2 \) and that \( \hat{f}(\xi) \) vanishes for almost every \( \xi < 0 \). First, we note that
\[
\hat{h}_b(\xi) = \lim_{R \to +\infty} \int_{-R}^R e^{-i\xi x} h(x + ib) dx
\]
so that for any sequence \( \{ R_j \} \) with \( R_j \to +\infty \) as \( j \to \infty \), there is a subsequence with the property that
\[
(D.1) \quad \hat{h}_b(\xi) = \lim_{j \to +\infty} \int_{-R_j}^{R_j} e^{-i\xi x} h(x + ib) dx
\]
for almost every \( \xi \). Let \( 0 < b_1 < b_2 \), let \( \{ R_j \} \) be a sequence (to be chosen later) with \( R_j \to \infty \), and \( j \to \infty \), and let \( \Gamma_j \) be the rectangular path consisting of the four segments
\[
\begin{align*}
\gamma_1(t) &= t + ib_1 & \quad & -R_j \leq t \leq R_j \\
\gamma_2(t) &= R_j + it & \quad & b_1 \leq t \leq b_2 \\
\gamma_3(t) &= -t + ib_2 & \quad & -R_j \leq t \leq R_j \\
\gamma_4(t) &= -R_j + i(b_1 + b_2 - t) & \quad & b_1 \leq t \leq b_2
\end{align*}
\]
We will show that \( h_{b_1}(\xi) = h_{b_2}(\xi) \) exploiting the fact that
\[
\int_{\Gamma_j} e^{-iz\xi} h(z) \, dz = 0
\]
is zero by analyticity and showing that \( \int_{\gamma_2} e^{-i\xi z} h(z) \, dz \to 0 \) and \( \int_{\gamma_4} e^{-i\xi z} h(z) \, dz \to 0 \) as \( j \to \infty \). To do this, let
\[
w(x) = \int_{b_1}^{b_2} e^{iy} h(x + iy) \, dy
\]
and note that \( w(R_j) = \int_{\gamma_2} e^{-i\xi z} h(z) \, dz \) and \( w(-R_j) = \int_{\gamma_4} e^{-i\xi z} h(z) \, dz \) (where \( R_j \) is still to be chosen!). By \( w \in L^2(\mathbb{R}) \) since
\[
\int_{-\infty}^{\infty} |w(x)|^2 \, dx \leq (b_2 - b_1) \left( \int_{b_1}^{b_2} e^{2y} \, dy \right) \sup_{b_1 \leq y \leq b_2} \left( \int_{-\infty}^{\infty} |h(x + iy)|^2 \, dx \right)
\]
so there is indeed a sequence \( \{ R_j \} \) with \( R_j \to \infty \) as \( j \to \infty \) so that \( w(R_j) \to 0 \) and \( w(-R_j) \to 0 \) as \( j \to \infty \). By passing to a subsequence if necessary we may also suppose that \( \hat{h}_{b_1}(\xi) = \lim_{j \to +\infty} \int_{-R_j}^{R_j} e^{-i\xi x} h(x + ib_1) dx \) and \( \hat{h}_{b_2}(\xi) = \lim_{j \to +\infty} \int_{-R_j}^{R_j} e^{-i\xi x} h(x + ib_2) dx \) for almost every \( \xi \). It now follows that \( e^{ib_1} h_{b_1}(\xi) = e^{ib_2} h_{b_2}(\xi) \) for almost every \( \xi \), and we define \( f \) by \( \hat{f}(\xi) = e^{ib\xi} \hat{h}_b(\xi) \). Since \( h_b \in L^2(\mathbb{R}) \) for all \( b > 0 \), it follows that \( \int_{-\infty}^{\infty} e^{-2b\xi} |\hat{f}(\xi)|^2 \, d\xi \) is bounded uniformly in \( b > 0 \) and so, by Fatou’s lemma, considering the behavior as \( b \downarrow 0 \), it follows that \( f \in L^2(\mathbb{R}) \).

\footnote{Recall that \( w = \lim_{R \to +\infty} \| w - g_R \|_{L^2(\mathbb{R})} \to 0 \) as \( R \to \infty \).}
On the other hand, \( \int_{-\infty}^{b} e^{-2b\xi} |\hat{f}(\xi)|^2 \, d\xi \) is also uniformly bounded as \( b \uparrow \infty \), from which it follows that \( \hat{f}(\xi) = 0 \) for almost every \( \xi < 0 \).

|APPENDIX E. COMPACT OPERATORS|

A bounded linear operator \( T \) from a Banach space \( X \) to itself is called **compact** if the image of any bounded sequence has a convergent subsequence (equivalently, \( T \) is compact if it maps bounded subsets of \( X \) to subsets of \( X \) with compact closure).

We review some of the basics of the Riesz-Schauder theory of compact operators: a good reference is chapter X.5 of Yosida’s book on functional analysis [4], whose treatment (theorems and proofs) we follow here. We will need the following simple Theorem of F. Riesz. Recall that if \( Y \) is a subspace of a normed linear space \( X \), then

\[
\text{dist} (x, Y) = \inf \{ \| x - y \| : y \in Y \}
\]

**Lemma E.1.** Let \( X \) be a normed linear space and let \( M \) be a proper closed linear subspace. For any 0 < \( \varepsilon < 1 \) there is an \( x_\varepsilon \in X \) with \( \| x_\varepsilon \| = 1 \) and \( \text{dist}(x_\varepsilon, M) \geq 1 - \varepsilon \).

**Proof.** Pick \( y \in X - M \). Since \( M \) is closed, \( \alpha := \text{dist}(y, M) > 0 \) strictly and there is an \( m_\varepsilon \in M \) with

\[
\| y - m_\varepsilon \| \leq \alpha \left( 1 + \frac{\varepsilon}{1 + \varepsilon} \right)
\]

Let \( x = (y - m_\varepsilon)/\| y - m_\varepsilon \| \). Then \( \| x \| = 1 \) and

\[
\| x - m \| = (\| y - m_\varepsilon \|)^{-1} (y - m_\varepsilon - m \| y - m_\varepsilon \|)
\geq \alpha^{-1} \left( 1 + \frac{\varepsilon}{1 - \varepsilon} \right)^{-1} \alpha
= 1 - \varepsilon
\]

\( \Box \)

If \( T \) is any bounded linear operator, \( \text{Ker} (\lambda I - T) \) is closed, but \( \text{Ran} (\lambda I - T) \) is not closed. However, for compact operators this is always the case.

**Lemma E.2.** Suppose that \( T \) is compact. For any \( \lambda \neq 0 \), \( \text{Ran} (\lambda I - T) \) is closed.

**Proof.** Without loss take \( \lambda = 1 \). Suppose that \( \{ y_n \} \) is a sequence from \( \text{Ran}(I - T) \), so that \( y_n = (I - T)x_n \). Let \( \alpha_n = \text{dist}(x_n, \text{Ker}(I - T)) \) and choose \( u_n \in \text{Ker}(I - T) \) with \( \alpha_n < \| x_n - u_n \| < \alpha_n(1 + 1/n) \). If \( \{ \alpha_n \} \) is bounded then the vectors \( w_n = x_n - u_n \) are a bounded sequence with \( y_n = (I - T)w_n \), so that there is a subsequence \( \{ w_{n'} \} \) so that \( Tw_{n'} \) converges. As \( w_n = y_n + Tw_n \) it follows that \( w_n \) converges to an element \( w \in X \) with \( y = (I - T)x \), so \( y \in \text{Ran}(I - T) \). If \( \alpha_n \to \infty \) as \( n \to \infty \), let \( z_n = w_n/\| w_n \| \) so that \( \| z_n \| = 1 \) and \( (I - T)z_n \to 0 \) as \( n \to \infty \). By compactness there is a subsequence \( \{ z_{n'} \} \) so that \( Tz_{n'} \) converges, so \( z_{n'} \) converges to an element \( z \) with \( (I - T)z = 0 \). Let \( q_n = z_n - z \). Then \( x_n - u_n - z \| w_n \| = q_n \| w_n \| \), so the right-hand side must have norm at least \( \alpha_n \) since \( u_n - z \| w_n \| \in \text{Ker}(I - T) \). Hence \( \| q_n \| \| w_n \| > \alpha_n \), a contradiction since \( \| w_n \| < \alpha_n(1 + 1/n) \) and \( \| q_n \| \to 0 \) as \( n \to \infty \). \( \Box \)

To prove the Fredholm alternative we will need the following corollary to the open mapping theorem.
Theorem E.3. Suppose that $X$ and $Y$ are Banach spaces and $A$ is a bounded linear mapping of $X$ onto $Y$ with trivial kernel. Then $A^{-1}$ exists as a bounded linear operator from $Y$ to $X$.

We now prove the main result of this Appendix:

Theorem E.4. (Fredholm Alternative) Let $X$ be a Banach space and suppose that $T : X \to X$ is compact. Then, either

(i) $\ker(I + T)$ is nonempty or
(ii) $(I + T)^{-1}$ exists.

Proof. It suffices to show that if $\ker(I + T)$ is empty, then $(I + T)^{-1}$ exists. From the Lemma, Ran$(I + T)$ is closed, so a corollary of the Open Mapping Theorem (see, for example, chapter II.5 of [4]), $(I + T)^{-1}$ is a bounded operator from Ran$(I + T)$ onto $X$. Thus, it suffices to prove that Ran$(I + T) = X$.

If not, then Ran$(I + T)$ is a proper closed subspace of $X$, and we claim that the sequence of closed spaces $X_n$ given by $X_n = (I + T)^n X$ satisfies $X_{n+1} \subset X_n$ (proper containment). To see this, first note that by Lemma E.1 there is a vector $y \in X$ with $\|y\| = 1$ and $\text{dist}(y, X_1) \geq 1/2$. We will prove inductively that $X_{n+1}$ is properly contained in $X_n$ for all $n$ by finding $y_n \in X_n$ with $\text{dist}(y_n, X_{n+1}) > 0$. Since $(I + T) : X_{n-1} \to X_n$ is onto with trivial kernel, it follows that $(I + T)^{-1}$ exists as a bounded operator from $X_n$ onto $X_{n-1}$ Let $y_n = (I + T)^{-1} y$. We claim that $\text{dist}(y_n, X_{n+1}) > 0$. If $x \in X_{n+1}$ then $x = (I + T) z$ for $z \in X_n$. As $y_{n-1} - z = (I + T)^{-1} (y_n - x)$ we have

$$\|y_n - x\| \geq \|(I + T)^{-1}\|^{-1} \|y_{n-1} - z\| \geq \|(I + T)^{-1}\|^{-1} \text{dist}(y_{n-1}, X_{n-1})$$

from which it follows that

$$\text{dist}(y_n, X_{n+1}) \geq \|(I + T)^{-1}\|^{-1} \text{dist}(y_{n-1}, X_{n})$$

proving the claim.

We will now derive a contradiction. The sequence $\{Ty_n\}$ has no convergent subsequence since, if $n > m$

$$T(y_n - y_m) = y_{n+1} - [y_n + y_{m+1} - y_m]$$

and $\text{dist}(y_{n+1}, X_n) \geq 1/2$. This contradicts the compactness of $T$. \hfill \Box

The Fredholm alternative is a very useful tool in the study of integral equations of the form $(I + T)x = y$ where $T$ is an integral operator. A useful paraphrase of the Fredholm alternative is: “if the solution is unique, it exists.” A key step in the analysis is to show that the operator $T$ is compact. It is often useful to consider integral operators acting on $L^p(\mathbb{R}^n)$ or $L^p(X)$ for a measurable subset of $\mathbb{R}^n$.

The following characterization of compact sets for $L^p(\mathbb{R})$, $1 \leq p < \infty$ (the Fréchet-Kolmogorov Theorem, e.g. [4], Chapter X.1) is useful for determining when integral operators are compact. It carries over with no essential change to $L^p(0, \infty)$.

Proposition E.5. Let $S$ be a bounded subset of $L^p(\mathbb{R})$. Then $S$ is compact if and only if
(1) For any $\varepsilon > 0$ there is a $\delta > 0$ so that $\int_{-\infty}^{\infty} |f(t+h) - f(t)|^p \, dt < \varepsilon$ for $|h| < \delta$, uniformly in $f \in S$, and

(2) For any $\varepsilon > 0$, there is an $R > 0$ so that $\int_{|x|>R} |f(t)|^p \, dt < \varepsilon$ uniformly in $f \in S$.

This should be compared to the Arzela-Ascoli Theorem which characterizes compact subsets of $C(X)$ where $X$ is a compact metric space. The first condition is an $L^p$-version of equicontinuity, while the second condition says that the set $S$ consists of functions which are, in some sense, uniformly localized. The proof of necessity uses the fact that, for any $\varepsilon > 0$, there is a finite set of functions $\{f_i\}_{i=1}^N$ so that $\|f - f_i\|_p < \varepsilon$ for some $i$. The proof of sufficiency uses the fact that compactly supported continuous functions are dense in $L^p$ and then uses the Arzela-Ascoli theorem.

References


