A SPECTRAL APPROACH TO CONSECUTIVE PATTERN-AVOIDING PERMUTATIONS

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Abstract. We consider the problem of enumerating permutations in the symmetric group on n symbols which avoid a given consecutive pattern S, and in particular computing asymptotics as n tends to infinity. We develop a general method which solves this enumeration problem using the spectral theory of integral operators on $L^2([0,1]^m)$, where the pattern S has length $m+1$. Kreĭn and Rutman's generalization of the Perron-Frobenius theory of nonnegative matrices plays a central role. Our methods give detailed asymptotic expansions and allow for explicit computation of leading terms in many cases.

1. Introduction

In this paper, we study integral operators of the form

$$(Tf)(x_1,\ldots,x_m) = \int_0^1 \chi(t,x_1,\ldots,x_m) f(t,x_1,\ldots,x_{m-1})\, dt \tag{1.1}$$

and their application to enumerating permutations that avoid a consecutive pattern. Here χ is a real-valued function on $[0,1]^m$ which takes values in $[0,1]$ and is continuous away from a set of measure zero in $[0,1]^{m+1}$. As we will show, operators of this type arise naturally when counting permutations that avoid a consecutive pattern of length $m+1$.

To define the enumeration problem, let $\mathfrak{S}_n$ denote the symmetric group on n elements. For $\pi \in \mathfrak{S}_n$ we write $\pi = (\pi_1\pi_2\cdots\pi_n)$ where the π_k are the integers from 1 to n. For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all distinct i and j, we denote by $\Pi(x)$ the unique permutation $\pi \in \mathfrak{S}_n$ with $\pi_i < \pi_j$ if and only if $x_i < x_j$. A *pattern* of length k is an element σ of $\mathfrak{S}_k$. If π is a permutation of length $n \geq k$, π avoids σ if $\Pi(\pi_j, \pi_{j+1}, \ldots, \pi_{j+k-1}) \neq \sigma$ for all j with $1 \leq j \leq n-k+1$. More generally, if $S \subseteq \mathfrak{S}_k$ we say that π avoids S if π avoids each $\sigma \in S$. That is, S is the set of forbidden patterns.

For a subset S of $\mathfrak{S}_{m+1}$, denote by $\alpha_n(S)$ the number of permutations $\pi \in \mathfrak{S}_n$ that avoid S. Observe that for $n \leq m$ we have $\alpha_n(S) = n!$. Our goal is to compute asymptotics of $\alpha_n(S)$ as n tends to infinity. Throughout the paper we will assume that $m \geq 2$.

For $S \subseteq \mathfrak{S}_{m+1}$ we define a function χ_S on $[0,1]^{m+1}$ by setting $\chi_S(x) = 0$ if $x_i = x_j$ for some distinct indices i and j, and otherwise setting

$$\chi_S(x) = \begin{cases} 1 & \Pi(x) \notin S \\ 0 & \Pi(x) \in S \end{cases} \tag{1.2}$$

Finally, let T_S be the operator of the form (1.1) with $\chi = \chi_S$. Let $(\,\cdot\,,\,\cdot\,)$ be the standard inner product on $L^2([0,1]^m)$, and let $\mathbf{1}$ denote the constant function 1

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on $[0,1]^m$. It is straightforward to prove (see Proposition 2.4) that the formula

$$\frac{\alpha_n(S)}{n!} = \left(\mathbf{1}, T_S^{n-m}\mathbf{1}\right) \tag{1.3}$$

holds. Note that the left-hand side is the probability of selecting a permutation $\pi \in \mathfrak{S}_n$ at random that avoids the set S.

The asymptotic behavior of powers of a bounded linear operator is determined by its spectrum. Recall that if A is a bounded linear operator from a Hilbert space to itself, the resolvent set of A is the set of all $z \in \mathbb{C}$ so that $(A - zI)^{-1}$ is also a bounded operator. The complement in $\mathbb{C}$ of the resolvent set is the spectrum of A, denoted $\sigma(A)$, and the spectral radius of A is given by

$$r(A) = \sup\left\{|\lambda| : \lambda \in \sigma(A)\right\}.$$

It is easy to show (see, for example. Theorem VI.6 of [15]) that

$$r(A) = \lim_{n \longrightarrow \infty} \left\|A^n\right\|^{1/n}.$$

The operator A is called a *Volterra operator* if $r(A) = 0$. If $r(A) \neq 0$, all of the eigenvalues of A are contained in the closed disc of radius $r(A)$ about zero. The peripheral spectrum of A, denoted $P(A)$, is the set of all $\lambda \in \sigma(A)$ with $|\lambda| = r(A)$. For the operators T, the peripheral spectrum consists of at most finitely many eigenvalues of finite multiplicity.

Let us write $\rho(S)$ for $r(T_S)$. Using spectral theory, we shall prove:

Theorem 1.1. *Let S be a nonempty set of forbidden patterns. Then $\rho(S) < 1$ strictly and*

$$\rho(S) = \lim_{n \longrightarrow \infty} \left(\frac{\alpha_n(S)}{n!}\right)^{1/n}$$

We give the proof of Theorem 1.1 in Section 2.1. This result extends Theorem 4.1 of [7], where consecutive patterns consisting of a single permutation were considered. Moreover, Theorem 1.1 characterizes the limit in terms of a spectral quantity which can be computed in many cases of interest by solving the eigenvalue problem for the integral operator T_S.

For some sets of patterns, we can show that the peripheral spectrum consists of a single, simple, positive eigenvalue and obtain leading asymptotics of $\alpha_n(S)/n!$ with spectral estimates on the remainder. A sufficient condition for the peripheral spectrum of T_S to consist only of the simple eigenvalue $\rho(S)$ may be formulated as follows. If (X, μ) is a measure space and $f \in L^2(X, d\mu)$ is a real-valued function, we will say that $f > 0$ if $f(x) > 0$ for almost every $x \in X$, and $f \geq 0$ if $f(x) \geq 0$ for almost every x. A bounded operator A on $L^2(X, \mu)$ is *positivity improving* if for any $f \geq 0$ different from 0 there is an integer k (possibly depending on f) so that $A^k f > 0$. Let A^* denote the Hilbert space adjoint of A. Kreĭn and Rutman [13] showed that, for such an operator, $P(A)$ consists exactly of the simple eigenvalue $r(A)$ and there is a normalized eigenfunction φ with $\varphi > 0$. Moreover, A^* also has a simple eigenvalue at $r(A)$ and an eigenfunction $\psi > 0$. Let us normalize ψ by the condition $(\psi, \varphi) = 1$. Setting $A = T_S$ and letting ψ and φ be the normalized eigenfunctions of T_S and T_S^* with eigenvalue $\rho(S)$, we then have:

Theorem 1.2. *Suppose that T_S is positivity improving. Then*

$$\frac{\alpha_n(S)}{n!} = (\psi, \mathbf{1})\,(\mathbf{1}, \varphi)\,\rho(S)^n + \mathcal{O}(\rho_1(S)^n).$$

A sufficient condition for T_S to be positivity improving can be formulated in combinatorial terms as follows. If S is a set of patterns let G_S be the graph with vertex set $\mathfrak{S}_m$ and a directed edge from π to σ if there is a permutation $\tau \in \mathfrak{S}_{m+1} \backslash S$ with $\Pi(\tau_1, \dots, \tau_m) = \pi$ and $\Pi(\tau_2, \dots, \tau_{m+1}) = \sigma$. The graph $G_\emptyset$ is known as the graph of overlapping permutations. The graph G_S is *strongly connected* if any vertex of G_S is connected to any other vertex by a directed path.

Theorem 1.3. *Suppose that the graph G_S is strongly connected and that the monotone permutations $12 \cdots m+1$ and $m+1\ m \cdots 1$ do not belong to S. Then the operator T_S is positivity improving.*

An application of this theorem is the following. Call a permutation π in $\mathfrak{S}_n$ *reducible* if there exists an index i such that $1 \le i \le n-1$ and $\pi(1), \dots, \pi(i) \le i$ (equivalently $i+1 \le \pi(i+1), \dots, \pi(n)$). A permutation that is not reducible is called *irreducible.*

Theorem 1.4. *Let S be a subset of $\mathfrak{S}_{m+1}$ such that each permutation in S is irreducible and and the monotone permutations $m+1\ m \cdots 1$ do not belong to S. Then $\alpha_n(S)$ has the form*

$$\frac{\alpha_n(S)}{n!} = c \cdot \rho^n + \mathcal{O}(\rho_1^n),$$

where c, ρ and ρ_1 are positive constants and $\rho > \rho_1$.

As a corollary we have the following result due to Elizalde [7]:

Corollary 1.5. *Let S consists of a single permutation. Then the asymptotics of $\alpha_n(S)$ is given by $c \cdot \rho^n \cdot n!$ where c and ρ are positive constants.*

More generally we can characterize the spectrum of T_S in terms of another graph associated to S. To define it, let Δ_π denote the set of points $x = (x_1, \dots, x_m) \in (0,1)^m$ with $x_i \neq x_j$ for $i \neq j$ and $\Pi(x) = \pi$. The graph H_S has vertex set $\cup_{\pi \in \mathfrak{S}_m} \Delta_\pi$ and directed edges from $(x_1, \dots, x_m)$ to $(x_2, \dots, x_{m+1})$ if $x_1 \neq x_{m+1}$ and $\Pi(x_1, \dots, x_{m+1}) \notin S$. We will say that the graph H_S is *strongly connected* if any two vertices x and y can be connected by a directed path and there is a universal upper bound on the length of the minimal path between any two points. We define the *period* of a strongly connected graph G as follows. Fix a vertex v of G and, for k a nonnegative integer, let X_k be the set of all vertices in G that can be reached from v in k steps. The set Q of all k with $v \in X_k$ is a semigroup and generates a subgroup $d\mathbb{Z}$ of $\mathbb{Z}$. The integer d is the *period* of the graph G. Note that, if G is strongly connected, then G has period d for some positive integer d. Finally, a graph is called *ergodic* if it strongly connected and has period 1.

Theorem 1.6. *Suppose that S is a nonempty set of forbidden patterns and that the graph H_S is strongly connected with period d. Then the operator T_S has spectral radius $\rho(S)$ with $0 < \rho(S) < 1$, and T_S has a simple eigenvalue $\lambda = \rho(S)$ with strictly positive eigenfunction. Moreover, the spectrum of T_S is invariant under multiplication by* $\exp(2\pi i/d)$.

In case $d = 1$ we conclude that the peripheral spectrum consists of only one simple, real, positive eigenvalue, and we obtain a sharper asymptotic formula than that of Theorem 1.1.

Theorem 1.7. *Suppose that S is a nonempty set of forbidden patterns and that the graph H_S is ergodic. Then*

$$\frac{\alpha_n(S)}{n!} = (\psi, \mathbf{1})\,(\mathbf{1}, \varphi)\,\rho(S)^n + \mathcal{O}(\rho_1(S)^n)$$

where

$$\rho_1(S) = \sup\{\lambda \in \sigma(T_S) : |\lambda| < \rho(S)\}$$

We can also distinguish the cases $\rho(S) = 0$ and $\rho(S) > 0$ in terms of the graph H_S.

Theorem 1.8. *Suppose that S is a nonempty set of forbidden patterns. Then $\rho(S) > 0$ if and only if the graph H_S has a directed cycle.*

Thus, if H_S has a directed cycle, then T_S must have a positive eigenvalue and strictly positive eigenfunction.

For certain explicit patterns, we can compute the spectrum and eigenfunctions of T_S and obtain sharp asymptotic formulas for $\alpha_n(S)/n!$.

Example 1.9. If $S = \{123\}$ we show that T_{123} is an operator with trivial kernel and spectrum given by $\{\lambda_k\}_{k=-\infty}^{\infty}$ where

$$\lambda_k = \frac{\sqrt{3}}{2\pi \cdot \left(k + \frac{1}{3}\right)}.$$

We also compute the eigenfunctions of T_{123} and T_{123}^* and show that

$$\frac{\alpha_n(123)}{n!} = \exp\left(\frac{1}{2\lambda_0}\right) \cdot \lambda_0^{n+1} + \mathcal{O}\left(|\lambda_{-1}|^n\right) \tag{1.4}$$

where $\lambda_0 = \rho(S)$. For more terms in the asymptotic expansion see Theorem 5.3.

Example 1.10. If $S = \{213\}$, we show that the nonzero eigenvalues of T_{213} are the roots of the equation

$$\operatorname{erf}\left(\frac{1}{\sqrt{2}\lambda}\right) = \sqrt{\frac{2}{\pi}}$$

which has the unique real root $\lambda_0 = 0.7839769312\ldots$. Moreover, λ_0 is the largest root of the equation. We then have

$$\frac{\alpha_n(213)}{n!} = \exp\left(\frac{1}{2\lambda_0^2}\right) \cdot \lambda_0^{n+1} + \mathcal{O}\left(|\lambda_1|^n\right) \tag{1.5}$$

where $\lambda_{1,2} = 0.2141426360\ldots \pm 0.2085807022\ldots \cdot i$ are the next two largest roots of the eigenvalue equation.

Example 1.11. If $S = \{123, 321\}$, the numbers $\alpha_n(S)$ are given by $\alpha_n(S) = 2E_n$ for $n \geq 2$ where E_n is the nth Euler number. In this case, we can use spectral methods to obtain the classical convergent expansion

$$\frac{E_n}{n!} = 2 \cdot \sum_{\substack{j \geq 1 \\ j \text{ odd}}} (-1)^{\frac{j-1}{2}(n+1)} \left(\frac{\pi j}{2}\right)^{-n-1}.$$

This formula was derived by Ehrenborg, Levin, and Readdy [6] (Corollary 4.2) by using Fourier series. In this case the spectrum of the operator T_S real and invariant under the reflection $\lambda \mapsto -\lambda$; in particular $P(S) = \{2/\pi, -2/\pi\}$. Moreover, the matrix U_S introduced in Proposition 2.3, is similar to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for a cyclic permutation of order two.

It is easy to find examples of patterns S for which $\rho(T_S) = 0$.

Example 1.12. Let $S = \{132, 231\}$. An S-avoiding permutation has no peaks (viewed as the graph of a function from $\{1, \ldots, n\}$ to itself) and it is easy to see that $\alpha_n(S) = 2^{n-1}$. However, observe that the proportion $\alpha_n(S)/n!$ is subexponential. It is straightforward to verify that the operator T_S has no non-zero eigenvalues. Also note that the graph G_S is not strongly connected.

Example 1.13. Let $S = \{123, 213, 231, 321\}$. The directed graph G_S is strongly connected but the monotone permutations 123 and 321 are excluded. In this case $\alpha_n(S) = 2$ for all $n \geq 2$.

We close our introduction by a brief overview on the subject of pattern avoidance in permutations (for more details we refer to [2]). The "classical" definition of a pattern is slightly different than one provided above. We say that a permutation π avoids a pattern σ if π does not contain a *subsequence* which is order-isomorphic to σ. The study of such patterns originated in theoretical computer science by Donald Knuth [12]. However, the first systematic study was done by Simon and Schmidt [16], who completely classified the avoidance of patterns of length three. Since then several hundred papers related to the field have been published.

One of the most important results in the subject is the proof by Marcus and Tardos [14] of the so-called Stanley-Wilf conjecture related to the asymptotic behavior of the number of permutations that avoid a given pattern. It states that for any permutation σ there exists a constant c (depending on σ) such that the number of the permutations of length n that avoid σ is less than c^n.

In this paper we also study asymptotic behavior of permutations avoiding patterns, but we consider *consecutive* patterns, occurrences of which correspond to (contiguous) factors, rather than subsequences, anywhere in permutations. Simultaneous avoidance of consecutive patterns of length 3 is studied in [11].

Suppose p is a consecutive pattern. Recall that $\alpha_n(p)$ denotes the number of n-permutations avoiding p. It is known [7] that $\lim_{n \longrightarrow \infty} \sqrt[n]{\alpha_n(p)/n!}$ is a nonnegative constant. Moreover, in [8] asymptotics for the following consecutive patterns is given: 123, 132, 1342, 1234 and 1243. These results are obtained by representation of permutations as increasing binary trees, then using symbolic methods followed by solving certain linear differential equations with polynomial coefficients to get corresponding exponential generating functions, and, finally, using the following result (see [9, Chapter 4] for a discussion).

Theorem 1.14. *Let $A(z)$ be a meromorphic function on a domain of the complex plane including the origin, and let ρ be the unique pole of $A(z)$ such that $|\rho|$ is minimum. Then the following asymptotic estimate holds:*

$$[z^n]A(z) \sim \gamma \cdot \rho^{-n}$$

where γ is the residue of A in ρ.

In our paper we develop a general method (not involving generating functions) that gives detailed asymptotic expansions and allows for explicit computation of leading terms in many cases. As special cases of our results, we get a more detailed asymptotics for some of the results of Elizalde and Noy [8]. We note that, in the case of consecutive patterns, their exponential generating function is related to our operator T_S by the formula

$$\sum_{n\geq 0}\alpha_n(S)\frac{z^n}{n!} = 1+\cdots+z^m+z^{m+1}(\mathbf{1},(I-zT_S)^{-1}\mathbf{1})$$

From this formula it is clear that the radius of convergence of the generating function is determined by the spectrum of T_S.

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2. The Operator T_S

We begin by considering integral operators of the form

$$(Tf)(x) = \int_0^1 \chi(t,x_1,\ldots,x_m)f(t,x_1,\ldots,x_{m-1})\ dt.$$

If T^* denotes the Hilbert space adjoint of T, then

$$\left(T^*f\right)(x) = \int_0^1 \chi(x_1,\ldots,x_m,u)f(x_2,\ldots,x_m,u)\ du$$

so that T^* is also an operator of the above form. For $n\geq m$, let

$$\chi_n(x_1,\ldots,x_n) = \prod_{j=1}^{n-m}\chi(x_j,\ldots,x_{m+j}) \tag{2.1}$$

If $k<m$ then

$$\left(T^kf\right)(x) = \int_{[0,1]^k}\chi_{m+k}(t_1,\ldots,t_k,x_1,\ldots,x_m)f(t_1,\ldots,t_k,x_1\ldots,x_{m-k})\ dt_1\cdots dt_k \tag{2.2}$$

while if $k\geq m$,

$$\left(T^kf\right)(x) = \int_{[0,1]^k}\chi_{m+k}(t_1,\ldots,t_k,x_1,\ldots,x_m)\ f(t_1,\ldots,t_m)\ dt_1\cdots dt_k \tag{2.3}$$

Lemma 2.1. *Let T be an operator of the form (1.1) with $0\leq\chi(t,x_1,\ldots,x_m)\leq 1$. Then $\|T\|\leq 1$, T^m is compact, and the nonzero spectrum of T consists of discrete eigenvalues of finite multiplicity which may accumulate only at 0.*

Proof. Applying the Cauchy-Schwarz inequality in the variable t we obtain

$$\begin{aligned}&\left(\int_0^1 \chi(t,x_1,\ldots,x_m)f(t,x_1,\ldots,x_{m-1})\ dt\right)^2\\ \leq&\int_0^1|\chi(t,x_1,\ldots,x_m)|^2\ dt\cdot\int_0^1|f(t,x_1,\ldots,x_{m-1})|^2\ dt\\ \leq&\int_0^1|f(t,x_1,\ldots,x_{m-1})|^2\ dt.\end{aligned}$$

Now integrating over $[0,1]^m$ we have $\|T(f)\|^2 \leq \|f\|^2$, proving that $\|T\| \leq 1$. The operator T^m takes the form

$$(T^m f)(x) = \int_{[0,1]^m} K(x,s)\ f(s)\ ds$$

with $K(x,s) = \chi_{2m}(s_m, \ldots, s_1, x_1, \ldots, x_m)$ which is a bounded function. It follows that T^m is a Hilbert-Schmidt operator, and hence compact. Thus the spectrum of T^m consists of discrete eigenvalues which may accumulate only at 0. Let ω be $\exp(2\pi i/m)$, that is, a primitive mth root of unity. The identity

$$I - A^m = \prod_{j=0}^{m-1} \left(I - \omega^j A\right), \tag{2.4}$$

applied to $A = T/\lambda$ for $\lambda \neq 0$ shows that $\lambda^m \in \sigma(A)$ if and only if $\lambda\omega^j \in \sigma(A)$ for some j. Thus, the nonzero spectrum of T consists of isolated points which may accumulate only at zero. Such isolated points must be eigenvalues of T, and using formula (2.4) again we can show that such eigenvalues have finite multiplicity. □

In Subsection 3.2 we give a condition in terms of the directed graph H_S when the spectral radius is positive. Now we consider in greater detail the case when the spectral radius is positive.

Proposition 2.2. *Let T be an operator of the form (1.1) and suppose that $r(T) > 0$. Then T has a positive eigenvalue $\rho = r(T)$ with strictly positive eigenfunction φ, and T^* has a positive eigenvalue ρ with strictly positive eigenfunction ψ.*

Proof. Since T preserves the cone of positive functions in $L^2([0,1]^m)$ and T has nonzero spectral radius, it follows from Theorem 6.1 of [13] and the fact that the cone of positive functions is self-dual that T and T^* both have $\rho = r(T)$ as an eigenvalue with at least one strictly positive eigenfunction. □

We may normalize φ and ψ by taking $\|\varphi\| = 1$ and $(\psi, \varphi) = 1$. We have, moreover, the following decomposition of T.

Proposition 2.3. *Let T be an operator of the form (1.1) and suppose that $r(T) > 0$. Then the operator T admits a decomposition of the form*

$$T = \rho U + W$$

where $UW = WU = 0$, U has finite-dimensional range, U maps the interior of the cone of positive functions into itself, $\rho(W) < \rho$ strictly, and the eigenvalues of U are roots of unity including 1.

Proof. The orthogonal decomposition follows from Theorem 8.1 of [13] applied to the operator $A = \rho^{-1}T$. □

In particular, the leading behavior of powers T^n is determined by ρ and the finite-rank operator U.

2.1. **Connection with Pattern Avoidance.** Now suppose $S \subseteq \mathfrak{S}_{m+1}$ and let T_S be an operator of the form (1.1) with

$$\chi(t, x_1, \ldots, x_m) = \chi_S(t, x_1, \ldots, x_m)$$

where χ_S is defined in (1.2). It is not difficult to see that, for this choice of χ and for any $n \geq m$, the function χ_n defined in (2.1) obeys

$$\chi_n(x) = \begin{cases} 1 \text{ if } \Pi(x) \text{ avoids } S \\ 0 \text{ if } \Pi(x) \text{ does not avoid } S \end{cases}$$

for $x = (x_1, \ldots, x_n)$ with $x_i \neq x_j$ for all distinct indices i and j. In particular, χ_n is continuous except on a set of measure 0 in $[0,1]^n$. It follows from the definition that

$$\int_{[0,1]^n} \chi_n(x)\ dx = \frac{\alpha_n(S)}{n!} \tag{2.5}$$

since each simplex in the standard triangulation of $[0,1]^n$ corresponds to a permutation $\pi \in \mathfrak{S}_n$ and each such simplex has volume $1/n!$. Using this observation and the identities (2.2) and (2.3), we can prove:

Proposition 2.4. *The formula (1.3) holds for any $n \geq m+1$, and $\rho(S) < 1$ for any nonempty set of forbidden patterns S.*

Proof. Formulas (2.2) and (2.3) show that for $k \geq 1$, $x \in [0,1]^m$, and $t \in [0,1]^k$

$$\left(T^k \mathbf{1}\right)(x) = \int_{[0,1]^k} \chi_{m+k}(t_1, \ldots, t_k, x_1, \ldots, x_m)\ dt_1 \cdots dt_k.$$

Since

$$\left(\mathbf{1}, T^k \mathbf{1}\right) = \int_{[0,1]^m} \left(T^k \mathbf{1}\right)(x)\ dx,$$

formula (1.3) follows. To see that $\|T_S^m\| < 1$, first take $n = 2m$ in (2.5) we have

$$\int_{[0,1]^{2m}} \chi_{2m}(t,x)\ dt\ dx = \frac{\alpha_{2m}(S)}{(2m)!} < 1$$

if S is a non-empty pattern. On the other hand, formula (2.3), the fact that $\chi_{2m}^2 = \chi_{2m}$, and the Cauchy-Schwarz inequality imply that

$$\left|\left(T^m f\right)(x)\right|^2 \leq \left(\int_{[0,1]^m} \chi\left(t,x\right)\ dt\right)\left(\int_{[0,1]^m} \left|f(t)\right|^2 dt\right)$$

which gives the estimate

$$\left\|T^m f\right\|^2 \leq \left(\frac{\alpha_{2m}(S)}{(2m)!}\right)\left\|f\right\|^2$$

on integration over x. This shows that $\|T_S^m\| < 1$. ¿From the inequality $\left\|T_S^{mq+r}\right\| \leq \left\|T_S^m\right\|^q \left\|T_S\right\|^r$ it follows that $\rho(S) = \lim_{n \longrightarrow \infty} \left\|T_S^n\right\|^{1/n} < 1$ strictly. □

Proof of Theorem 1.1. We have already shown that $\rho(S) < 1$ in Proposition 2.4. Suppose first that $\rho(S) = 0$. ¿From the inequality $|(\mathbf{1}, T_S^n \mathbf{1})| \leq \|T_S^n\|$ we immediately conclude that $\left(\alpha_n(S)/n!\right)^{1/n}$ tends to 0 as n goes to infinity. If $\rho(S) > 0$, then by Propositions 2.2, 2.3, and 2.4 we have

$$\frac{\alpha_n(S)}{n!} = \rho(S)^n\left(\mathbf{1}, U^n \mathbf{1}\right) + \left(\mathbf{1}, W^n \mathbf{1}\right)$$

where the second term obeys the estimate

$$\left|\left(\mathbf{1}, W^n \mathbf{1}\right)\right| \leq \left(\rho(S) - \varepsilon\right)^n$$

for some $\varepsilon > 0$ and all sufficiently large n. Moreover, $(\mathbf{1}, U^n \mathbf{1})$ is periodic in n and strictly positive since U maps the interior of the cone of positive functions into itself. Thus $(\mathbf{1}, U^n \mathbf{1})$ is both bounded and bounded below by a strictly positive constant. It follows that $\lim_{n \longrightarrow \infty} (\mathbf{1}, U^n \mathbf{1})^{1/n} = 1$ so that $\lim_{n \longrightarrow \infty} (\alpha_n(S)/n!)^{1/n} = \rho(S)$ as claimed. □

2.2. **Eigenvalue expansion.** In order to find an asymptotic expansion of the quantity $\alpha_n(S)/n! = \left(\mathbf{1}, T_S^{n-m}\mathbf{1}\right)$ we use the spectral theory of the operator T. In this section we will only assume that T^m is compact and that T has non-trivial spectrum.

Let λ be non-zero and belonging to the spectrum of T. Then λ is an isolated singularity of T. Hence $\ker(T - \lambda I)$ is nonempty since $\ker(T - \lambda I)$ is the range of the projection given by the residue of $(T - zI)^{-1}$ at $z = \lambda$. Since any eigenvector of T with eigenvalue λ is also an eigenvector of T^m with eigenvalue λ^m and T^m is compact, it follows that $V_\lambda = \ker(T - \lambda I)$ has finite dimension N_λ for any $\lambda \neq 0$.

Now consider the adjoint operator T^*. Since $(T^*)^m$ is compact it follows that $\sigma(T^*)$ is a discrete set whose only accumulation point is 0. It is not difficult to see that $\sigma(T^*)\setminus\{0\}$ consists of those λ with $\overline{\lambda} \in \sigma(T)$. A similar argument applies to T^*, and the identity

$$\left[(T - zI)^{-1}\right]^* = (T^* - \overline{z}I)^{-1}$$

shows that the finite-dimensional space $W_\lambda = \ker(T^* - \overline{\lambda}I)$ has the same dimension as V_λ. Recall that

$$P_\lambda = \operatorname*{Res}_{z=\lambda} (T - zI)^{-1}$$

projects onto V_λ, so clearly P_λ^* projects onto W_λ.

Now let $\left\{\varphi^i\right\}_{i=1}^{N_\lambda}$ be an orthogonal basis for V_λ. By the Riesz representation theorem, the functional $\psi \mapsto \left(\varphi^i, P_\lambda(\psi)\right)$ is represented by a vector ψ_i so that

$$P_\lambda = \sum_{i=1}^{N_\lambda} (\psi^i, \ \cdot\ )\varphi^i.$$

Since P_λ^* is the projection onto $\ker\left(T^* - \overline{\lambda}I\right)$, the vectors ψ_i are eigenvectors of T^* with eigenvalue $\overline{\lambda}$. The condition that $P_\lambda^2 = P_\lambda$ implies that

$$\left(\psi^i, \varphi^j\right) = \delta_{i,j} \tag{2.6}$$

These conditions suffice to determine the ψ^j given a choice of $\left\{\varphi^j\right\}$.

We now consider the spectral expansion of T^n, assuming now that $\sigma(T)$ is included in an open disc of radius R. ¿From the analytic functional calculus we have

$$T^n = \frac{1}{2\pi i} \int_{|z|=R} (T - zI)^{-1} \ z^n \ dz$$

Now pick $r > 0$ such that (i) all the eigenvalues $\{\lambda_j\}_{j=1}^k$ lie in the exterior of the disc of radius r and (ii) the circle $|z| = r$ contains no eigenvalues of T. This choice is possible since $\sigma(T)$ is discrete. Let P_j denote the projection map P_{λ_j}. By shrinking the contour to a circle of radius r and taking into account the poles at the eigenvalues $\lambda_1, \ldots, \lambda_k$ we have

$$T^n = \sum_{j=1}^{k} \lambda_j^n P_j + \frac{1}{2\pi i} \int_{|z|=r} (T - zI)^{-1} z^n dz.$$

We bound this integral by

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{|z|=r} (T - zI)^{-1} z^n dz \right\| &\leq \frac{1}{2\pi} \int_{|z|=r} \left\| (T - zI)^{-1} dz \right\| \cdot r^n \\ &\leq \sup_{|z|=r} \left\| (T - zI)^{-1} \right\| \cdot r^n \\ &= \mathcal{O}(r^n). \end{aligned}$$

We can now summarize this discussion as:

Theorem 2.5. *Let T be an operator such that T^m is compact, Let r be a positive real number such that there is no eigenvalue of T with modulus r. Furthermore let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of T with that are greater in modulus than r. Then we have*

$$(\mathbf{1}, T^n(\mathbf{1})) = \sum_{j=1}^{k} c_j \lambda_j^n + \mathcal{O}(r^n)$$

where

$$\begin{aligned} c_j &= (\mathbf{1}, P_j(\mathbf{1})) \\ &= \sum_{i=1}^{N_j} \left(\psi_j^i, \mathbf{1}\right) \left(\mathbf{1}, \varphi_j^i\right) \end{aligned}$$

where $\left\{\varphi_j^i\right\}$ and $\left\{\psi_j^i\right\}$ are bases for the $\lambda = \lambda_j$ eigenspaces of T and T^*, respectively, so chosen that the normalization (2.6) holds.

In case when the operator T has no nonzero eigenvalues, Theorem 2.5 is not useful. All it says is that $(\mathbf{1}, T^n(\mathbf{1}))$ grows subexponential.

3. Associated graphs

3.1. **The directed graph H_S.** In this section we study the spectrum of T_S using the infinite graph H_S described in the introduction. Let Δ_π° denote the open subset of $(0,1)^m$ with $x_i \neq x_j$ for $i \neq j$ and $x_i < x_j$ if and only if $\pi(i) < \pi(j)$, and let

$$X = \bigcup_{\pi \in \mathfrak{S}_m} \Delta_\pi^\circ$$

Thus $(0,1)^m - X$ is a set of measure zero consisting of those x with $x_i = x_j$ for at least one pair of distinct indices i and j. The graph H_S has vertex set X. Recall that the directed edges of H_S connected points x and y in X with $x_{j+1} = y_j$ for $1 \leq j \leq m-1$, $x_1 \neq y_m$, and $\Pi(x_1, \ldots, x_m, y_m) \notin S$. It follows from the definition that $\chi(t, x_1, \ldots x_m) = 1$ if and only if there is a directed edge from $(t, x_1, \ldots, x_{m-1})$ to $(x_1, \ldots, x_m)$. That is, the function χ encodes the edge information of the graph H_S.

The next lemma connects the graph H_S to mapping properties of the operator T_S. In the remainder of the is section we write T for T_S.

Lemma 3.1. *Suppose that $x, y \in X$ and that there is a directed path from x to y of length $k \geq m$. Suppose further that f is a nonnegative continuous function such that f is non-zero in a neighborhood of x. Then $(T^k f)(y) > 0$.*

Proof. Let the directed path be $(x_1, \ldots, x_m)$, $(x_2, \ldots, x_{m+1})$, …, $(x_{k+1}, \ldots, x_{k+m})$ where $y = (x_{k+1}, \ldots, x_{k+m})$. Let ε be the minimum of the following finite set

$$\{|x_i - x_j| : 1 \leq i < j \leq k+m,\ j - i \leq m\} \cup \{x_i, 1 - x_i : 1 \leq i \leq k+m\}.$$

Observe that $\varepsilon > 0$ by the definition of X. Let $\delta = \varepsilon/3$. For $s_i \in [x_i - \delta, x_i + \delta]$, $1 \leq i \leq k+m$, we have that $(s_1, \ldots, s_m)$, $(s_2, \ldots, s_{m+1})$, …, $(s_{k+1}, \ldots, s_{k+m})$ is also a directed path in H_S. It follows that $\chi_{k+m}(s_1, \ldots, s_{k+m}) = 1$ for all such s. Using (2.3) we may estimate

$$\begin{aligned}
\left(T^k f\right)(y) &\geq \int_{x_1-\delta}^{x_1+\delta} \cdots \int_{x_k-\delta}^{x_k+\delta} f(t_1, \ldots, t_m)\ dt_1 \cdots dt_k \\
&= (2\delta)^{k-m} \int_{x_1-\delta}^{x_1+\delta} \cdots \int_{x_m-\delta}^{x_m+\delta} f(t_1, \ldots, t_m)\ dt_1 \cdots dt_m \\
&> 0
\end{aligned}$$

where in the last step we have used the positivity of f in a neighborhood of $(x_1, \ldots, x_m)$. ☐

Proposition 3.2. *Suppose that H_S is strongly connected with period d. Then there is a decomposition*

$$X = \bigcup_{i=0}^{d-1} Y_i$$

of X into disjoint sets Y_i with the property that $T : L^2(Y_i) \longrightarrow L^2(Y_{i+1})$, where $Y_d = Y_0$.

Proof. Pick a base vertex v of H_S. Let X_k be the set of all vertices in H_S that can be reached from v in k steps, and let Q be the subset of the nonnegative integers defined by

$$Q = \{k : v \in X_k\}.$$

Then Q is a semigroup under addition and generates a subgroup of the integers $\mathbb{Z}$. A subgroup of $\mathbb{Z}$ has the form $d\mathbb{Z}$ for some positive integer d; in this case, d is the period of the graph H_S.

Now define

$$Y_i = \bigcup_{j:\ j \equiv i \bmod d} X_j.$$

Observe that every directed edge in the graph H_S goes from some Y_i to the next Y_{i+1} (with addition modulo d). Also, observe that the sets Y_i are pairwise disjoint.

We claim that each Y_i is open. To see this, suppose that $y \in Y_i$. Pick a path from some vertex x to the vertex y having length greater than $2m$. We can perturb this path in a small neighborhood of y using a variant of the argument used at the beginning of Lemma 3.1 and conclude that Y_i is open

Now pick a permutation $\pi \in \mathfrak{S}_m$. Since the sets $\Delta_\pi^\circ \cap Y_i$ are all open and Δ_π° is connected, it follows for a given permutation, at most one of the intersections $\Delta_\pi^\circ \cap Y_i$ is nonempty. Thus each Y_i is the disjoint union of interiors of standard simplices.

Finally, suppose that f is a continuous function on $[0,1]^m$ with support in one of the Y_i. We claim that Tf is supported in Y_{i+1}. To see this, note that $\chi(t, x_1, \ldots, x_m) = 1$ if and only if there is a directed edge from $(t, x_1, \ldots, x_{m-1})$ to $(x_1, \ldots, x_m)$. Thus, if f is a continuous function supported in Y_i. $(Tf)(x) \neq 0$ only if x is connected to a point of Y_i by a directed path of length one, i.e., $x \in Y_{i+1}$. It follows that Tf is supported in Y_{i+1}. □

Proof of Theorem 1.6. It follows from Proposition 3.2 that $T^d : L^2(Y_i) \longrightarrow L^2(Y_i)$. We will denote by A the restriction of T^d to $L^2(Y_0)$. Choosing a positive integer p with $pd \geq m$ we see that A^p is a compact operator from $L^2(Y_0)$ to itself. The operator A^p has discrete spectrum which may accumulate only at 0 so the same is true of A. We claim that, if φ is an eigenvector of A with nonzero eigenvalue λ^d, then $T^i\varphi$ is a nonzero eigenvector of T^d with eigenvalue λ^d and support in Y_i. The only nontrivial part of this claim is that $T^i\varphi \neq 0$. To see this, note that $T^{d-i}(T^i\varphi) = \lambda^d\varphi$ so $T^i\varphi$ cannot be zero since $\lambda \neq 0$ and φ is a nontrivial eigenfunction.

Suppose now that $\lambda^d \in \sigma(A)$, let φ be an eigenvector of A corresponding to the eigenvalue λ^d, let ω be a dth root of unity, and let

$$\psi = \varphi + \frac{1}{\lambda\omega}T\varphi + \cdots + \left(\frac{1}{\lambda\omega}\right)^{d-1} T^{d-1}\varphi.$$

The vector ψ is nonzero because the right-hand terms are nonzero and have disjoint supports. A direct computation shows that $T\psi = \omega\lambda\psi$ so the spectrum of T contains all of the numbers $\lambda\omega$ where $\lambda^d \in \sigma(A)$. On the other hand, any eigenvalue μ of T gives rise to an eigenvalue μ^d of T^d, so the nonzero spectrum of T consists exactly of the numbers $\omega\lambda$ where ω is a dth root of unity and λ^d is an eigenvalue of T^d. □

3.2. **Cycles in the graph H_S.** We now prove that the operator T_S has positive spectral radius if and only if the graph H_S has a directed cycle. The proof has two parts, one graph-theoretic and one analytic. First, we prove that if H_S has a single directed cycle, there is a neighborhood of directed cycles. This condition is equivalent to the condition that for some $k \geq m$, the integral kernel of T_S^k is nonzero in a set of positive measure that intersects the diagonal in a set of positive measure (we give a more precise formulation below). Second, we show that this analytic condition on the integral kernel holds if and only if T_S has nonzero spectral radius.

The first step is:

Lemma 3.3. *Assume that $(x_1, \ldots, x_m)$, $(x_2, \ldots, x_{m+1})$, …, $(x_{k+1}, \ldots, x_{k+m}) = (x_1, \ldots, x_m)$ is a directed cycle of length $k \geq m$ in the graph H_S. Then there exists a $\delta > 0$ such that for all s_i in the interval $[x_i - \delta, x_i + \delta]$ that $(s_1, \ldots, s_m)$, $(s_2, \ldots, s_{m+1})$, …, $(s_{k+1}, \ldots, s_{k+m}) = (s_1, \ldots, s_m)$ is a directed cycle of lenght k.*

Proof. The proof is similar to Lemma 3.1. Pick ε by the same procedure as in Lemma 3.1 and let $\delta = \varepsilon/3$. □

To carry out the second step we will use several ideas from the theory of trace-class operators which must be modified for our setting. We begin with a simpler case which illustrates the main ideas. Suppose that A is an integral operator on a closed subset X of $\mathbb{R}^n$, i.e.,

$$(Af)(x) = \int_X K(x,y)f(y)\,dy$$

where $K(x,y)$ is continuous. In this case, A and all of its powers are trace-class and one can define a determinant

$$D(z) = \det(I - zA)$$

using the formal identity

$$\log D(z) = \operatorname{Tr}\log(I - zA)$$

leading to

$$D(z) = \exp\left(-\sum \frac{z^n}{n}\operatorname{Tr}(A^n)\right)$$

It can be shown that $D(z)$ is an entire function whose zeros ζ_n are related to the nonzero eigenvalues of A by $\zeta_n = 1/\lambda_n$. In particular, A has zero spectral radius if and only if $D(z) = 1$, i.e., if $\operatorname{Tr}(A^n) = 0$ for all n. If K is nonnegative, then the integral kernels $K^{(n)}$ of the trace-class operators A^n are also nonnegative, and $D(z) = 1$ if and only if $K^{(n)}(x,x) = 0$ for all $x \in X$ and each $n \geq 1$.

To analyze our case, this strategy must be modified since (1) the operator T_S is not trace-class, but rather T_S^k is trace-class if $k \geq 2m$ and (2) the integral kernel of T_S and its powers is not continuous, but only piecewise constant.

Suppose that X is a bounded measurable subset of $\mathbb{R}^k$ and let $T : L^2(X) \to L^2(X)$ be a trace-class operator with integral kernel $K(x,y) \in L^2(X \times X)$. We extend K to a function on $\mathbb{R}^k \times \mathbb{R}^k$ by setting $K(x,y) = 0$ if $(x,y) \notin X \times X$ and define

$$\widetilde{K}(x,y) = \lim_{r \downarrow 0} \frac{1}{(2r)^{2k}} \int_{[-r,r]^k} \int_{[-r,r]^k} K(x+s, y+t)\, ds\, dt$$

The limit exists for almost every $(x,y) \in X \times X$. Brislawn [3] shows that if K is the integral kernel of a trace-class operator, then $\widetilde{K}(x,x)$ exists for almost every x and

$$\operatorname{Tr}(T) = \int_X \widetilde{K}(x,x)\, dx.$$

One does not expect the diagonal $\widetilde{K}(x,x)$ to be well-defined for an arbitrary function in $L^2(X \times X)$; the hypothesis that K be the integral kernel of a trace-class operator is quite strong and Brislawn's proof of this fact and the trace formula use the explicit eigenfunction expansion for the integral kernel that results from the facts that T is a trace-class operator.

For a trace-class operator T, the determinant

$$D(\lambda) = \prod_{j-1}^{\infty} (1 - \lambda\lambda_j)$$

where $\{\lambda_j\}$ are the eigenvalues of T, ordered so that $|\lambda_{j+1}| \leq |\lambda_j|$. Thus T has spectral radius zero if and only if $D(\lambda) = 1$ for all λ. We also have the expansion

$$D(\lambda) = \exp\left[\operatorname{Tr}\log(I - \lambda T)\right]$$

where for $|\lambda|$ small we have

$$\operatorname{Tr}\log(I - \lambda T) = -\sum_{j=1}^{\infty} \frac{\lambda^j}{j}\operatorname{Tr}\left(T^j\right) \tag{3.1}$$

From these considerations, we easily see:

Proposition 3.4. *Let X be a bounded measurable subset of $\mathbb{R}^n$, let $T : L^2(X) \to L^2(X)$ be a trace-class operator with Hilbert-Schmidt integral kernel K, and suppose that $K(x,y) \geq 0$ a.e. Let $K^{(j)}$ be the integral kernel of T^j for $j \geq 1$. Then T has nonzero spectral radius if and only if for some $m \geq 1$, $\widetilde{K^{(j)}}(x,x) \neq 0$ on a set of nonzero measure.*

Proof. Let $\sigma_j = \operatorname{Tr}\left(T^j\right)$ and note that, as $K(x,y) \geq 0$ a.e., the same is true of $K^{(j)}$ and hence $\widetilde{K^{(j)}}$. It follows that the numbers σ_j are all nonnegative. Since $\log D(\lambda) = -\sum_{j=1}^{\infty} \sigma_j \lambda^j / j$ it is clear that $D(\lambda) = 1$ if and only if $\sigma_j = 0$ for all j. This in turn holds if and only if $\widetilde{K^{(j)}}(x,x) = 0$ for a.e. x. and each j. □

It will be convenient to have a version of Proposition 3.4 which applies to operators in the ideal $\mathcal{I}_p$ of bounded operators T with T^p a trace-class operator (here we assume that p is an integer although the $\mathcal{I}_p$-classes are defined for any $p > 0$). We recall from [18] that if $T \in \mathcal{I}_p$, the renormalized determinant

$$D_p(\lambda) = \prod_{j=1}^{\infty} f_p(\lambda \lambda_j)$$

where

$$f_p(z) = (1 - z) \exp\left(\sum_{j=1}^{p-1} \frac{1}{j} z^j\right)$$

defines an analytic function of λ with zeros at $1/\lambda_j$, and $D_p(\lambda) = 1$ iff T has no nonzero eigenvalues. In analogy to (3.1) we have

$$\log D_p(\lambda) = -\sum_{j=p}^{\infty} \frac{\lambda^j}{j} \operatorname{Tr}\left(A^j\right)$$

for small λ.

Proposition 3.5. *Let X be a bounded measurable subset of $\mathbb{R}^n$, let $T : L^2(X) \to L^2(X)$ be a bounded operator, and let $p \geq 1$ be a positive integer so that T^p is a trace-class operator with Hilbert-Schmidt integral kernel K. Finally, suppose that $K(x,y) \geq 0$ a.e. and let $K^{(j)}$ be the integral kernel of T^j for $j \geq p$. Then T has nonzero spectral radius if and only if for some $j \geq p$, $\widetilde{K^{(j)}}(x,x) \neq 0$ on a set of nonzero measure.*

Proof. Essentially the same as before since

$$\log D_p(\lambda) = -\sum_{j=p}^{\infty} \sigma_j \frac{\lambda^j}{j}$$

□

Next, we consider the role of cycles in the graph H_S.

Proof of Theorem 1.8. One direction is easy (I think!): if H_S has a directed cycle, then we can show that there is a set of nonzero measure on which $K^{(j)}(x,y)$ is nonzero in a neighborhood of the diagonal. This means that the renormalized determinant has at least one zero and hence T has at least one nonzero eigenvalue. Where I'm stuck is the other direction: if one of the integral kernels $\tilde{K}^j(x,y)$ is nonzero on the diagonal for a set of nonzero measure, why then must there be a directed cycle? The fact that all of the functions involved are nonnegative (indeed averages of functions which are either zero or one!) should play a role. □

3.3. **The directed graph G_S.** We examine relations between the infinite graph H_S and the finite graph G_S. We need the following lemmas, the first one is a straightforward consequence of the definition of the graph H_S.

Lemma 3.6. *Suppose that $\alpha : (0,1) \longrightarrow (0,1)$ is a strictly increasing function. If $(x_1, \ldots, x_m)$, $(x_2, \ldots, x_{m+1})$, ..., $(x_{k+1}, \ldots, x_{k+m})$ is a directed path in H_S, then so is*

$$(\alpha(x_1), \ldots, \alpha(x_m)), (\alpha(x_2), \ldots, \alpha(x_{m+1})), \ldots, (\alpha(x_{k+1}), \ldots, \alpha(x_{k+m})).$$

Lemma 3.7. *Let $x = (x_1, \ldots, x_m)$ be a vertex in H_S such that $\Pi(x_1, \ldots, x_m) = \pi$. Furthermore assume there is an edge in G_S labeled τ leaving the vertex π. Then there exists x_{m+1} in $(0,1)$ such that $\Pi(x_1, \ldots, x_{m+1}) = \tau$. That is, there is an edge in H_S leaving the vertex x.*

Proof. Observe that $\tau(m+1)$ is bigger than exactly $\tau(m+1)-1$ of the numbers $\tau(1), \ldots, \tau(m)$. Hence pick x_{m+1} such that it is bigger than exactly $\tau(m+1)-1$ of the numbers $x_1, \ldots, x_m$. □

Iterating this lemma we obtain:

Lemma 3.8. *Given a directed path from π to σ in the graph G_S. Let x be a vertex in H_S such that $\Pi(x) = \pi$. Then this directed path can be lifted to a directed path in H_S that ends with a vertex y such that $\Pi(y) = \sigma$.*

Now we give a sufficient condition for H_S to be strongly connected in terms of the graph G_S.

Proposition 3.9. *Let $S \subseteq \mathfrak{S}_{m+1}$, suppose that G_S is strongly connected, and suppose that the two monotone permutations $1\ 2 \cdots m+1$ and $m+1\ m\ \cdots 1$ do not belong to the set S. Then the graph H_S is ergodic.*

Proof. We first prove that H_S is strongly connected. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be two vertices of H_S, and let $\pi = \Pi(x)$ and $\sigma = \Pi(y)$. Since G_S is strongly connected we can find a directed path from π to $m \cdots 21$ in G_S. This directed path lifts to a directed path from x to $z = (z_1, \ldots, z_m)$ in H_S, where $z_1 > z_2 > \cdots > z_m$.

For any directed graph G define the reverse graph G^* to the the graph G where we reverse the direction of each edge. In the reverse graph G_S^* we have a directed path from σ to $12 \cdots m$. Lifting this directed path to a directed path in H_S^* and then reversing the path, we obtain a directed path from $w = (w_1, \ldots, w_m)$ to y in H_S, where $w_1 < \cdots < w_m$.

Finally, there is a directed path from $m \cdots 21$ to $12 \cdots\ m$ in G_S. Hence we know there is a directed path from $u = (u_1, \ldots, u_m)$ to $v = (v_1, \ldots, v_m)$ in H_S, where $u_1 > \cdots > u_m$ and $v_1 < \cdots < v_m$.

Choose α so that $0 < \alpha < \min(z_m, w_1)$. We then have a directed path from $\alpha \cdot u$ to $\alpha \cdot v$. Now, there is a directed path from z to $\alpha \cdot u$ of length m, namely

$$z,\ (z_2, \ldots, z_m, \alpha u_1),\ \ldots\ ,\ (z_m, \alpha u_1, \ldots, \alpha u_{m-1}),\ \alpha \cdot u$$

using the fact that $(m+1) \cdots 21$ is not forbidden. We can now concatenate these five directed paths to obtain a path from x via z, via $\alpha \cdot u$, via $\alpha \cdot v$, via w, to y.

Since G_S is strongly connected and has $m!$ vertices, an upper bound on the length of the directed paths in G_S chosen above is $m!-1$. Hence, the path that we have constructed has length at most $3(m!-1)+2m$.

To observe that H_S has period 1 note that we can construct a directed path from the vertex x to the vertex y that has length one more than the above construction. Namely, the path from π to $m \cdots 21$ can extended it by adding the loop $m{+}1\ m\ \cdots 1$ at the end. Now by concatenating these two paths with a path from y to x we obtain two cycles whose lengths differ by one. Since the greatest common divisor of two consecutive integers is one, the graph H_S is ergodic. □

Recall that a permutation π in $\mathfrak{S}_n$ is *irreducible* if there is no index i such that $1 \leq i \leq n-1$ and $\pi(1), \ldots, \pi(i) \leq i$. Otherwise the permutation is reducible.

Proposition 3.10. *Let $S \subseteq \mathfrak{S}_{m+1}$ such that each permutation $\tau \in S$ is irreducible. Then the graph G_S is strongly connected.*

Proof. Given two vertices $\pi = (\pi_1, \ldots, \pi_m)$ and $\sigma = (\sigma_1, \ldots, \sigma_m)$ of G_S. Since the integers $\pi_1, \ldots, \pi_m, \sigma_1 + m, \ldots, \sigma_m + m$ are distinct, the following path (described by its edges) is well defined:

$$\Pi(\pi_1, \ldots, \pi_m, \sigma_1{+}m), \Pi(\pi_2, \ldots, \pi_m, \sigma_1{+}m, \sigma_2{+}m), \ldots, \Pi(\pi_m, \sigma_1{+}m, \ldots, \sigma_m{+}m),$$

This path goes from the vertex π to vertex σ. Note that every edge τ' on the path is reducible. Hence this path avoids the forbidden irreducible edges of S. □

Proof of Theorem 1.4. This follows directly from Theorem 1.2 and Propositions 3.9 and 3.10. □

Proof of Corollary 1.5. Let τ be the single permutation in the set S. If τ is one of the two monotone permutations the result will follow from descent pattern avoidance, see Theorem 4.2. If τ is irreducible the result follows from Theorem 1.4. Finally if τ is reducible apply Theorem 1.4 to the upside down permutation $(m+2-\tau(1), \ldots, m+2-\tau(m+1))$ which is not reducible. □

Note that there are examples of patterns S so that H_S does not have cycles even though the graph G_S has cycles.

Example 3.11. $S = \{312, 321\}$. In this case the graph G_S has a cycle. However, the graph H_S does not have a cycle and hence is not strongly connected. We observe this by noting that for a directed edge $(x,y) \longrightarrow (y,z)$ in H_S we have that $x < \max(y,z)$. Hence none of x_i's in a k-cycle $(x_1,x_2) \longrightarrow (x_2,x_3) \longrightarrow \cdots \longrightarrow (x_k,x_1) \longrightarrow (x_1,x_2)$ can be the largest.

Via the classical bijection $\pi \longmapsto \hat{\pi}$ (see [19, Section 1.3]) one obtains that the number $\{312, 321\}$-avoiding permutations are in bijection with involutions, that is, permutations π such that $\pi^2 = \mathrm{id}$. This was first observed by Claesson [4]. It follows that the generating function is $\exp(z + z^2/2)$ and the asymptotic is $1/\sqrt{2} \cdot \exp(-1/4) \cdot (n/e)^{n/2} \cdot \exp(\sqrt{n})$.

On the other hand, if G_S does not have a cycle, then neither does H_S

Lemma 3.12. *Let $S \subseteq \mathfrak{S}_{m+1}$ and suppose that G_S has a directed cycle that contains the two vertices $12 \cdots m$ and $m \cdots 21$. Moreover, assume that the two monotone permutations $12 \cdots m+1$ and $m+1 \cdots 21$ do not belong to the set S. Then the graph H_S contains a directed cycle.*

Proof. Pick a vertex $x = (x_1, \ldots, x_m)$ such that $\Pi(x) = 12\cdots m$. The directed path from the vertex $12\cdots m$ to the vertex $m\cdots 21$ can be lifted to a path in H_S from the vertex x to a vertex $y = (y_1, \ldots, y_m)$ where $\Pi(y) = m\cdots 21$. Similarly, pick a vertex $z = (z_1, \ldots, z_m)$ such that $\Pi(z) = m\cdots 21$. The directed path from the vertex $m\cdots 21$ to the vertex $12\cdots m$ can be lifted to a path in H_S from the vertex z to a vertex $w = (w_1, \ldots, w_m)$ where $\Pi(w) = 12\cdots m$.

Using the two monotone functions $\alpha, \beta : (0,1) \longrightarrow (0,1)$ defined by $\alpha(x) = (x+1)/2$ and $\beta(x) = x/2$, we have two directed paths: one from $\alpha(x)$ to $\alpha(y)$ and one from $\beta(z)$ to $\beta(w)$.

Using that $m+1\cdots 21$ is an edge in G_S we have the following path path from $\alpha(y)$ to $\beta(z)$, namely $\alpha(y) = (\alpha(y_1), \ldots, \alpha(y_m))$, $(\alpha(y_2), \ldots, \alpha(y_m), \beta(z_1))$, $\ldots$, $(\beta(z_1), \ldots, \beta(z_m)) = \beta(z)$. Similarly, we have the directed path $\beta(w) = (\beta(w_1), \ldots, \beta(w_m))$, $(\beta(w_2), \ldots, \beta(w_m), \alpha(x_1))$, $\ldots$, $(\alpha(x_1), \ldots, \alpha(x_m)) = \alpha(x)$.

Concatenate these four directed paths and we obtain a directed cycle in H_S. □

Thus we obtain:

Corollary 3.13. *Suppose that G_S does not have a cycle. Then $\rho(S) = 0$.*

4. Computational techniques

In this section we discuss descent pattern avoidance which is a special case of pattern avoidance. First we introduce an analogue of the de Bruijn graph D_U, which has the advantage that it is smaller than the graph G_S. Moreover, if the graph D_U is ergodic so is the graph $H_{S(U)}$ and we obtain that the associated operator is positivity improving. Second, for descent pattern avoidance we obtain that the eigenfunctions has a simplified form. Finally, in the last subsection we consider pattern avoidance that has symmetry. In these cases we show that we can obtain the adjoint eigenfunctions from the eigenfunctions.

4.1. **Descent pattern avoidance.** The descent set of a permutation π in the symmetric group on n elements is the subset of $\{1, \ldots, n-1\}$, given by $\{i : \pi_i > \pi_{i+1}\}$. An equivalent notion is the descent word, defined as follows. The descent word of the permutation π is the word $u(\pi) = u_1 \cdots u_{n-1}$ where $u_i = a$ if $\pi_i < \pi_{i+1}$ and $u_i = b$ otherwise.

Let U be a collection of ab-words of length m. The permutation π avoids the set U if there is no consecutive subword of the descent word of π contained in the collection U.

Descent pattern avoidance is a special case of consecutive pattern avoidance. For instance, permutations avoiding the word aab is the permutations avoiding the set $S = \{1243, 1342, 2341\}$, since these three permutations are the permutations with descent word aab. More formally, for U a subset of $\{a,b\}^m$ define $S(U) \subseteq \mathfrak{S}_{m+1}$ by

$$S(U) = \{\pi \in \mathfrak{S}_{m+1} \ : \ u(\pi) \in U\}.$$

Then the set of permutation avoiding the descent words in U is the set of permutations avoiding $S(U)$.

For U a subset of $\{a,b\}^m$ define the associated *de Bruijn graph* D_U by letting the vertex set be $\{a,b\}^{m-1}$. For $x, y \in \{a,b\}$ and $u \in \{a,b\}^{m-2}$ such that $xuy \notin U$ let there be a directed edge from xu to uy. When the set U is empty, the graph D_U is the classical de Bruijn graph D_{m-1}.

Lemma 4.1. *Let U be a subset of $\{a,b\}^m$. If there is a cycle c of length $N \geq n+1$ that do not consists only of the loop a^m or not only of the loop b^m, then the cycle c can be lifted to a cycle of length N in the graph $H_{S(U)}$.*

Proof. The cycle c goes through the vertices

$$v_1 v_2 \cdots v_{m-1},\ v_2 v_3 \cdots v_m,\ \ldots,\ v_N v_1 \cdots v_{m-2},$$

where each v_i is either a or b, and the indices are modulo N. We would like to pick N real numbers $x_1, \ldots, x_N$ in the interval $(0,1)$ such that

$$\begin{array}{lll} x_i < x_{i+1} & \text{if} & v_i = a, \\ x_i > x_{i+1} & \text{if} & v_i = b, \end{array} \tag{4.1}$$

where all the indices are modulo N. Since all the letters v_1 through v_N are not the same, we may assume that $v_{N-1} = a$ and $v_N = b$. Pick x_1 arbitrarily. Pick x_2 through x_{N-1} such that inequality (4.1) is satisfied. Finally, pick x_N in the interval $(\max(x_{N-1}, x_1), 1)$. Now in the graph $H_{S(U)}$ we have the cycle

$$(x_1, x_2, \ldots, x_m),\ (x_2, x_3, \ldots, x_{m+1}),\ \ldots,\ (x_N, x_1, \ldots, x_{m-1}).$$

□

Theorem 4.2. *Let U be a subset of $\{a,b\}^m$. If the de Bruijn graph D_U is strongly connected, then the graph $H_{S(U)}$ is also strongly connected. Furthermore, the de Bruijn graph D_U has the same period as the graph $H_{S(U)}$.*

Proof. Note that the graph D_U has the directed edge $a^{m-1}b$ since otherwise it would not be strongly connected. Given two vertices x and y in $H_{S(U)}$. To prove that $H_{S(U)}$ is strongly connected it is enough to find a directed path from x to y.

We can find a path from $u(\Pi(x))$ to a^{m-1} in the graph D_U that consists of at least $m+1$ edges. Similarly to the lifting lemma, Lemma 3.8, we can lift this path to a path in $H_{S(U)}$ that starts at the vertex x and ends, say, in the vertex $z = (z_1, \ldots, z_m)$. Note that $z_1 < \cdots < z_m$. Moreover, we can find a path from $a^{m-2}b$ to $u(\Pi(y))$ in D_U that has length at least $m+1$. Lift this path to a path that ends in the vertex y at begins at $w = (w_1, \ldots, w_m)$, where $w_1 < \cdots < w_{m-1} > w_m$.

Let $v_i = \max(z_i, w_{i-1})$ for $2 \leq i \leq m$. Observe that we have the string of inequalities $z_1 < v_2 < \cdots < v_m > w_m$. We can now concatenate these two paths as follows. Replace each occurrence of z_i and w_{i-1} by v_i for $2 \leq i \leq m$ in each of the two paths. Then we may connect the vertex $(z_1, v_2, \ldots, v_m)$ with the vertex $(v_2, \ldots, v_m, w_m)$ via the edge that goes across the edge with descent word $a^{m-1}b$. Thus the graph $H_{S(U)}$ is strongly connected.

Since there is a graph homomorphism from $H_{S(U)}$ to D_U we know that the period of D_U divides the period of $H_{S(U)}$. To see that the periods are equal, pick a vertex w of D_U that differs from a^{m-1} and b^{m-1}. Then any cycle of length greater than $m+1$ through the vertex w in D_U lifts to a cycle of the same length in $H_{S(U)}$. Hence the greatest common divisor of lengths of cycles through w is a multiple of the greatest common divisor of the cycle lengths of $H_{S(U)}$. Hence the two periods are equal.

Note that this argument only works when $m \geq 3$ since there is no such vertex w in the $m = 2$ case. But the remaining $m = 2$ case is straightforward to check. □

4.2. **Invariant subspace for descent pattern avoidance.** For an ab-word u of length $m-1$ define the descent polytope P_u to be the subset of the unit cube $[0,1]^m$ corresponding to all vectors with descent word u. That is,

$$P_u = \{(x_1, \ldots, x_m) \in [0,1]^m \ : \ x_i \leq x_{i+1} \text{ if } u_i = a \text{ and } x_i \geq x_{i+1} \text{ if } u_i = b\}.$$

Observe that the m-dimensional unit cube is the union of the 2^{m-1} descent polytopes P_u. Now the operator T corresponding to the descent pattern avoidance of the set U has the following form. For an ab-word u of length $m-2$ and $y \in \{a, b\}$ we have

$$T(f)|_{P_{uy}} = \int_0^{x_1} \chi(auy) \cdot f(t, x_1, \ldots, x_{m-1})|_{P_{au}} \ dt + \int_{x_1}^1 \chi(buy) \cdot f(t, x_1, \ldots, x_{m-1})|_{P_{bu}} \ dt, \tag{4.2}$$

where by abuse of notation we let $\chi(w) = 1$ if w does not belong to the set U and $\chi(w) = 0$ otherwise.

Proposition 4.3. *Let T be the operator associated with a descent pattern avoidance and k is an integer such that $1 \leq k \leq m-1$. Let u be an ab-word of length $m-1$. Then the function $T^k(f)$ restricted to the descent polytope P_u only depends on the variables x_1 through x_{m-k}.*

Proof. Proof by induction on k. When $k = 1$ observe that there is no variable x_m on the right hand side of equation (4.2). When $k \geq 2$, but still $k \leq m-1$, we know by induction that $T^{k-1}(f)$ do not depend on $x_{m-k}, \ldots, x_m$. By the shift of variables in the right hand side of equation (4.2), we then obtain that $T^k(f)$ do not depend on $x_{m-k-1}, \ldots, x_m$. □

Corollary 4.4. *Let T be the operator associated with a descent pattern avoidance and let φ be an eigenfunction associated with a non-zero eigenvalue λ. Let u be an ab-word of length $m-1$. Then the eigenfunction restricted to the descent polytope P_u only depends on the variable x_1.*

Proof. Since $\lambda^{m-1} \cdot \varphi = T^{m-1}(\varphi)$ the eigenfunction has the required form. □

Let V be the subspace of $L^2([0,1]^m)$ consisting of all functions f that only depend on the variable x_1 when restricted to each of the descent polytopes P_u. Observe that the subspace V is invariant under the operator T. That is, the operator T restricts to the subspace V. Moreover the constant function $\mathbf{1}$ belongs to V. Hence to understand the behavior of $T^n(\mathbf{1})$ it is enough to study this restricted operator.

In order to describe the subspace V more explicitly define for an ab-word u of length $m-1$ the polynomial $f(u; x_1)$ as follows:

$$f(u; x_1) = \int_{(x_1, x_2, \ldots, x_m) \in P_u} 1 \ dx_2 \cdots dx_m.$$

These polynomials was first introduced and studied in [6], with different indexing.

Let p be a vector $(p_u(x_1))_{u \in \{a,b\}^{m-1}}$. That is, the vector p consists of one-variable functions in the variable x_1 and is indexed by ab-words of length $m-1$. Consider the function f on $[0,1]^m$ defined by

$$f(x_1, \ldots, x_m)|_{P_u} = p_u(x_1)$$

for all ab-words u of length $m-1$. Observe that the function f belongs to $L^2([0,1]^m)$, and hence to the invariant subspace V, if and only if

$$\int_0^1 f(u;x_1) \cdot |p_u(x_1)|^2 \;\; dx_1 < \infty$$

for all ab-words u of length $m-1$. For two functions f and g in the subspace V, corresponding to the two vectors $(p_u(x_1))_{u\in\{a,b\}^{m-1}}$ and $(q_u(x_1))_{u\in\{a,b\}^{m-1}}$, the inner product is given by

$$(f,g) = \sum_{u\in\{a,b\}^{m-1}} \int_0^1 f(u;x_1)\cdot p_u(x_1)\cdot q_u(x_1) dx_1 .$$

We end this section by a structural result about the subspace V.

Proposition 4.5. *The invariant subspace V is isometrically isomorphic to the Hilbert space $L^2\left([0,1]^{2^{m-1}}\right)$.*

Proof. The isomorphism of the Hilbert spaces $V \longrightarrow L^2([0,1])^{2^{m-1}}$ is given by

$$(p_u(x_1))_{u\in\{a,b\}^{m-1}} \longmapsto \left(\sqrt{f(u;x_1)}\cdot p_u(x_1)\right)_{u\in\{a,b\}^{m-1}} .$$

□

4.3. **Symmetries.** Let J and R be the following two involutions on the space $L^2([0,1]^m)$:

$$\begin{aligned}(Jf)(x_1,x_2,\ldots,x_m) &= f(1-x_m,\ldots,1-x_2,1-x_1),\\ (Rf)(x_1,x_2,\ldots,x_m) &= f(x_m,\ldots,x_2,x_1).\end{aligned}$$

Observe that both J and R are self adjoint operators.

Lemma 4.6. *Assume that χ has the symmetry*

$$\chi(x_1,x_2,\ldots,x_m,x_{m+1}) = \chi(1-x_{m+1},1-x_m,\ldots,1-x_2,1-x_1).$$

Then the adjoint of the associated operator T is given by $T^* = JTJ$. Moreover, if φ is an eigenfunction of the operator T with eigenvalue λ then $J\varphi$ is an eigenfunction of the adjoint T^* with the eigenvalue λ.

Proof. We have that

$$\begin{aligned} JTJf(x_1,x_2,\ldots,x_m) &= JTf(1-x_m,\ldots,1-x_2,1-x_1)\\ &= J\int_0^1 f(1-x_{m-1},\ldots,1-x_1,1-t)\cdot\chi(t,x_1,\ldots,x_m)dt\\ &= \int_0^1 f(x_2,\ldots,x_m,1-t)\cdot\chi(t,1-x_m,\ldots,1-x_1)dt\\ &= \int_0^1 f(x_2,\ldots,x_m,t)\cdot\chi(1-t,1-x_m,\ldots,1-x_1)dt\\ &= \int_0^1 f(x_2,\ldots,x_m,t)\cdot\chi(x_1,\ldots,x_m,t)dt\\ &= T^*f(x_1,\ldots,x_{m-1},x_m).\end{aligned}$$

For the second statement consider the following line of equalities $T^*J\varphi = JTJJ\varphi = JT\varphi = \lambda J\varphi$. □

Similarly to Lemma 4.6 we have the next lemma. Its proof is similar to the previous proof and hence omitted.

Lemma 4.7. *Assume that* χ *has the symmetry*

$$\chi(x_1, x_2, \ldots, x_m, x_{m+1}) = \chi(x_{m+1}, x_m, \ldots, x_2, x_1).$$

Then we have that the adjoint of the associated operator T *is given by* $T^* = RTR$. *Moreover, if* φ *is an eigenfunction of the operator* T *with eigenvalue* λ *then* $R\varphi$ *is an eigenfunction of the adjoint* T^* *with the eigenvalue* λ.

5. 123-Avoiding Permutations

A 123-avoiding permutation is a permutation $\pi \in \mathfrak{S}_n$ with no index j so that $\pi_j < \pi_{j+1} < \pi_{j+2}$, where $1 \leq j \leq n-2$. Let $\alpha_n(123)$ denote the number of 123-avoiding permutations in $\mathfrak{S}_n$.

5.1. Eigenfunctions and Eigenvectors.

Since 123-avoiding permutations can be viewed as permutations with no double descents Corollary 4.4 allows us to recast then problem of finding eigenfunctions in two variables into finding two one-variable functions.

Proposition 5.1. *The eigenvalues* λ_k *of the operator* T *on* $L^2([0,1]^2)$ *are given by*

$$\lambda_k = \frac{\sqrt{3}}{2\pi \cdot \left(k + \frac{1}{3}\right)}, \tag{5.1}$$

where $k \in \mathbb{Z}$ *and the associated eigenfunctions* $\varphi_k = \begin{cases} p_k(x) & \text{if } 0 \leq x \leq y \leq 1 \\ q_k(x) & \text{if } 0 \leq y \leq x \leq 1 \end{cases}$ *are given by*

$$\varphi_k = \exp\left(-\frac{x}{2\lambda}\right) \cdot \begin{cases} \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \cdot \frac{x}{\lambda}\right) & \text{if } 0 \leq x \leq y \leq 1, \\ \sin\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \cdot \frac{x}{\lambda}\right) & \text{if } 0 \leq y \leq x \leq 1. \end{cases} \tag{5.2}$$

Proof. The defining equations for eigenvalues and eigenfunctions reduces to the integral system:

$$\lambda \cdot p(x) = \int_x^1 q(t)dt, \tag{5.3}$$

$$\lambda \cdot q(x) = \int_0^x p(t)dt + \int_x^1 q(t)dt. \tag{5.4}$$

First, differentiating with respect to x, we obtain the first-order system

$$\lambda \cdot p'(x) = -q(x), \tag{5.5}$$

$$\lambda \cdot q'(x) = p(x) - q(x). \tag{5.6}$$

These equations have only the trivial solution if $\lambda = 0$, so $\lambda = 0$ is not an eigenvalue. If $\lambda \neq 0$ then the first-order system (5.5)–(5.6) implies the second-order equation

$$\lambda^2 \cdot p''(x) + \lambda \cdot p'(x) + p(x) = 0.$$

This equation has the general solution

$$p(x) = A \cdot \exp\left(\frac{\omega}{\lambda} \cdot x\right) + B \cdot \exp\left(\frac{\omega^2}{\lambda} \cdot x\right), \tag{5.7}$$

where $\omega = \exp\left(\frac{2\pi i}{3}\right)$. That is, ω satisfies the relation $\omega^2 + \omega + 1 = 0$. Moreover, equation (5.5) implies that

$$q(x) = -\omega \cdot A \cdot \exp\left(\frac{\omega}{\lambda} \cdot x\right) - \omega^2 \cdot B \cdot \exp\left(\frac{\omega^2}{\lambda} \cdot x\right). \tag{5.8}$$

Setting $x = 0$ and $x = 1$ in equations (5.3) and (5.4) and using that $\lambda \neq 0$ we obtain the boundary conditions:

$$p(0) = q(0), \tag{5.9}$$

$$p(1) = 0. \tag{5.10}$$

Substituting the expressions for $p(x)$ and $q(x)$ from equations (5.5) and (5.6) into boundary condition (5.9) we obtain $A + B = -\omega \cdot A - \omega^2 \cdot B$. This is equivalent to $\omega \cdot A + B = 0$. Hence we may set $A = 1/2 \cdot \exp\left(\frac{\pi i}{6}\right)$ and $B = \overline{A} = 1/2 \cdot \exp\left(-\frac{\pi i}{6}\right)$. Substituting equation (5.5) into the second boundary condition (5.10) implies that

$$A \cdot \exp\left(\frac{\omega}{\lambda}\right) = -B \cdot \exp\left(\frac{\omega^2}{\lambda}\right) = \omega \cdot A \cdot \exp\left(\frac{\omega^2}{\lambda}\right).$$

Cancelling A on both sides and taking the logarithm gives

$$\frac{\omega}{\lambda} = \frac{2\pi i}{3} + \frac{\omega^2}{\lambda} + 2\pi i \cdot k,$$

where k is an integer. Since $\omega - \omega^2 = \sqrt{3} \cdot i$ we obtain expression (5.1). Moreover $p(x)$ is given by

$$\begin{aligned} p(x) &= 1/2 \cdot \exp\left(\frac{\pi i}{6}\right) \cdot \exp\left(\omega \cdot \frac{x}{\lambda}\right) + 1/2 \cdot \exp\left(-\frac{\pi i}{6}\right) \cdot \exp\left(\omega^2 \cdot \frac{x}{\lambda}\right) \\ &= \frac{\exp\left(-\frac{x}{2\lambda}\right)}{2} \cdot \left(\exp\left(\frac{\pi i}{6} + \frac{\sqrt{3}}{2} \cdot i \cdot \frac{x}{\lambda}\right) + \exp\left(-\frac{\pi i}{6} - \frac{\sqrt{3}}{2} \cdot i \cdot \frac{x}{\lambda}\right)\right) \\ &= \exp\left(-\frac{x}{2\lambda}\right) \cdot \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \cdot \frac{x}{\lambda}\right). \end{aligned}$$

Now equation (5.5) implies the claimed expression for $q(x)$. □

Note that the eigenvalues are ordered by

$$\lambda_0 > -\lambda_{-1} > \lambda_1 > -\lambda_{-2} > \lambda_2 > -\lambda_{-3} > \lambda_3 > \cdots > 0.$$

By applying the involution J we obtain the adjoint eigenfunction

$$\psi_k = \exp\left(\frac{y-1}{2\lambda}\right) \cdot \begin{cases} \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \cdot \frac{1-y}{\lambda}\right) & \text{if } 0 \leq x \leq y \leq 1, \\ \sin\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \cdot \frac{1-y}{\lambda}\right) & \text{if } 0 \leq y \leq x \leq 1. \end{cases} \tag{5.11}$$

Proposition 5.2. *For the eigenfunctions $\varphi_k = \varphi$ of T and $\psi_k = \psi$ of T^* with eigenvalue $\lambda_k = \lambda = \sqrt{3}/(2\pi(k+1/3))$,*

$$(\mathbf{1}, \varphi) = (\psi, \mathbf{1}) = \frac{\sqrt{3}}{2}\lambda^2 \tag{5.12}$$

$$(\psi, \varphi) = \frac{3}{4}(-1)^k \lambda \exp\left(-\frac{1}{2\lambda}\right) \tag{5.13}$$

In particular

$$\frac{(\psi, \mathbf{1})\,(\mathbf{1}, \varphi)}{(\psi, \varphi)} = (-1)^k \lambda^3 \exp\left(\frac{1}{2\lambda}\right) \tag{5.14}$$

Proof. Note that $(\mathbf{1}, \psi) = (\mathbf{1}, J\varphi) = (J\mathbf{1}, \varphi) = (\mathbf{1}, \varphi)$ since J is self-adjoint and $J\mathbf{1} = \mathbf{1}$. In the following calculations we use the facts that $\cos(\sqrt{3}/(2\lambda)) = (-1)^k/2$ and $\sin(\sqrt{3}/(2\lambda)) = (-1)^k\sqrt{3}/2$. We use the formulas (5.2) but drop the subscripts on p, q, and λ. First, we note that

$$\begin{aligned}(\mathbf{1}, \varphi) &= \int_{0 \le x \le y \le 1} p(x)\ dx\ dy + \int_{0 \le y \le x \le 1} q(x)\ dx\ dy \\ &= \int_0^1 (1-x)p(x)\ dx + \int_0^1 xq(x)\ dx\end{aligned}$$

Explicit computation shows that

$$\begin{aligned}\int_0^1 (1-x)p(x)\ dx &= \frac{\sqrt{3}}{2} \cdot \lambda^2 \cdot (1 - (-1)^k \cdot \exp(-1/(2\lambda))) \\ \int_0^1 xq(x)\ dx &= \frac{\sqrt{3}}{2} \cdot \lambda^2 \cdot (-1)^k \cdot \exp(-1/(2\lambda))\end{aligned}$$

which shows (5.12). Next, using (5.2) and (5.11) and dropping subscripts as before, we have

$$\begin{aligned}(\psi, \varphi) &= \int_{0 \le x \le y \le 1} p(x)p^*(y)\ dx\ dy + \int_{0 \le y \le x \le 1} q(x)q^*(y)\ dx\ dy \\ &= \int_0^1 \left(p(x) \cdot \int_x^1 p(1-y)dy + q(x) \cdot \int_0^x q(1-y)dy\right) dx.\end{aligned}$$

Carrying out the y integration and simplifying, we obtain

$$(\psi, \varphi) = \frac{3}{4}(-1)^k \int_0^1 \lambda \exp\left(-\frac{1}{2\lambda}\right)\ dx$$

which gives (5.13). □

5.2. **Asymptotics.** The above computations show that all eigenvalues of T_{123} are simple and given explicitly. We thus obtain the following expansion for $\alpha_n(123)/n!$ as an immediate consequence of Theorem 2.5, Propositions 5.1 and 5.2.

Theorem 5.3. *Let K be a nonnegative integer. The number of 123-avoiding permutations satisfies the following asymptotic expansion*

$$\frac{\alpha_n(123)}{n!} = \sum_{|k| \le K} (-1)^k \exp\left(\frac{1}{2\lambda_k}\right) \lambda_k^{n+1} + \mathcal{O}\left(r_{K+1}^n\right),$$

where λ_k is given by (5.1), $r_k = |\lambda_{-k}| = \sqrt{3}/2\pi \cdot \left(k - \frac{1}{3}\right)$ and the sum contains $2K+1$ terms corresponding to the $2K+1$ largest eigenvalues.

6. 213-Avoiding Permutations

A 213-avoiding permutation is a permutation $\pi \in \mathfrak{S}_n$ which contains no sequence of the form

$$\pi_{j+1} < \pi_j < \pi_{j+2}$$

for any j with $1 \leq j \leq n-2$. We denote the number of 213-avoiding permutations of $\mathfrak{S}_n$ by $\alpha_n(213)$. Thus, S consists of the single permutation 213 and

$$\chi_S(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_2 \leq x_1 \leq x_3, \\ 1 & \text{otherwise.} \end{cases}$$

By symmetry, the study of 213-avoiding permutations is equivalent to 132-avoiding permutations, 231-avoiding permutations and 312-avoiding permutations. However the case of 213-avoiding permutations gives the most straightforward equations.

6.1. Eigenfunctions and Eigenvectors.

In what follows, we will make use of the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\ dt \tag{6.1}$$

which extends to an entire function on $\mathbb{C}$, and the function

$$q(x) = \exp\left(-\frac{x^2}{2\lambda^2}\right). \tag{6.2}$$

Let

$$f(x,y) = \begin{cases} p(x,y) & \text{if } 0 \leq x \leq y \leq 1, \\ q(x,y) & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Then

$$(Tf)(x,y) = \begin{cases} \int_0^x p(t,x)dt + \int_y^1 q(t,x)dt & \text{if } 0 \leq x \leq y \leq 1, \\ \int_0^x p(t,x)dt + \int_x^1 q(t,x)dt & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Now we characterize the nonzero eigenvalues and eigenfunctions.

Proposition 6.1. *The non-zero eigenvalues λ of the operator T satisfies the equation*

$$\operatorname{erf}\left(\frac{1}{\sqrt{2}\cdot\lambda}\right) = \sqrt{\frac{2}{\pi}} \tag{6.3}$$

and the corresponding eigenfunctions are

$$\varphi(x,y) = \begin{cases} q(x) - \frac{1}{\lambda}\int_x^y q(t)\ dt & \text{if } x \leq y, \\ q(x) & \text{if } x > y, \end{cases}$$

where $q(x)$ is given by (6.2).

Proof. The defining relations for the eigenfunctions are

$$\lambda \cdot p(x,y) = \int_0^x p(t,x)dt + \int_y^1 q(t,x)dt, \tag{6.4}$$

$$\lambda \cdot q(x,y) = \int_0^x p(t,x)dt + \int_x^1 q(t,x)dt. \tag{6.5}$$

Now observe that in the right-hand side of equation (6.5) there is no dependency on the variable y. Hence we may replace $q(x,y)$ with $q(x)$. Now subtract equation (6.5) from equation (6.4)

$$\lambda \cdot (p(x,y) - q(x)) = -\int_x^y q(t)dt.$$

That is,

$$p(x,y) = q(x) - \frac{1}{\lambda}\int_x^y q(t)dt. \tag{6.6}$$

Substitute equation (6.6) into equation (6.5):

$$\begin{aligned}\lambda \cdot q(x) &= \int_0^x \left(q(t) - \frac{1}{\lambda}\int_t^x q(s)ds\right) dt + \int_x^1 q(t)dt \\ &= \int_0^1 q(t)dt - \frac{1}{\lambda}\int_0^x \int_t^x q(s)dsdt \\ &= \int_0^1 q(t)dt - \frac{1}{\lambda}\int_0^x \int_0^s q(s)dtds \\ &= \int_0^1 q(t)dt - \frac{1}{\lambda}\int_0^x s \cdot q(s)ds.\end{aligned}$$

Hence we have the following integral equation for $q(x)$

$$\lambda^2 \cdot q(x) = \lambda \cdot \int_0^1 q(t)dt - \int_0^x s \cdot q(s)ds. \tag{6.7}$$

Differentiating once we have

$$\lambda^2 \cdot q'(x) = -x \cdot q(x). \tag{6.8}$$

The solution to this differential equation is

$$q(x) = C \cdot \exp\left(-\frac{x^2}{2 \cdot \lambda^2}\right). \tag{6.9}$$

By setting the constant C to be 1 we obtain a solution to equation (6.8). Now substitute this solution for $q(x)$ into the integral equation (6.7) and set $x = 0$:

$$\begin{aligned}\lambda^2 &= \lambda \cdot \int_0^1 \exp\left(-\frac{t^2}{2 \cdot \lambda^2}\right) dt \\ &= \sqrt{2} \cdot \lambda^2 \cdot \int_0^{1/(\sqrt{2}\lambda)} \exp\left(-u^2\right) du \\ &= \frac{\sqrt{\pi} \cdot \lambda^2}{\sqrt{2}} \cdot \operatorname{erf}(1/(\sqrt{2}\lambda)),\end{aligned}$$

where the substitution in the integral is $u = t/(\sqrt{2}\lambda)$. Hence the non-zero eigenvalues λ satisfies equation (6.3). ∎

For completeness we state:

Lemma 6.2. *The eigenfunction of the operator T associated with eigenvalue 0 has the form*

$$\varphi(x,y) = \begin{cases} p(x,y) & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{if } 0 \le y \le x \le 1. \end{cases}$$

where $p(x,y)$ satisfies $\int_0^x p(t,x) = 0$.

The adjoint operator T^* is given by

$$T^*(f(x,y)) = \begin{cases} \int_0^y q(y,u)du + \int_y^1 p(y,u)du & \text{if } 0 \le x \le y \le 1, \\ \int_0^y q(y,u)du + \int_y^x p(y,u)du & \text{if } 0 \le y \le x \le 1. \end{cases}$$

Proposition 6.3. *For a non-zero eigenvalues λ of the operator T the corresponding eigenfunction of the adjoint operator T^* is*

$$\psi(x,y) = \begin{cases} p^*(y) & \text{if } 0 \le x \le y \le 1, \\ p^*(y) - \frac{1}{\lambda} \cdot \int_x^1 p^*(u)du & \text{if } 0 \le y \le x \le 1. \end{cases}$$

where

$$p^*(y) = -2 \cdot y \cdot \exp\left(\frac{y^2}{2\lambda^2}\right) + 2 \cdot \lambda + \sqrt{2\pi} \cdot y \cdot \exp\left(\frac{y^2}{2\lambda^2}\right) \cdot \text{erf}\left(\frac{y}{\sqrt{2}\lambda}\right). \tag{6.10}$$

Proof. The defining relations for the eigenfunctions are

$$\lambda \cdot p^*(x,y) = \int_0^y q^*(y,u)du + \int_y^1 p^*(y,u)du, \tag{6.11}$$

$$\lambda \cdot q^*(x,y) = \int_0^y q^*(y,u)du + \int_y^x p^*(y,u)du. \tag{6.12}$$

Observe that there is no dependency on the variable x in equation (6.11). Thus we write $p^*(x,y) = p^*(y)$. Subtracting these two equations we have

$$\lambda \cdot (q^*(x,y) - p^*(y)) = -\int_x^1 p^*(u)du,$$

such that

$$q^*(x,y) = p^*(y) - \frac{1}{\lambda} \cdot \int_x^1 p^*(u)du. \tag{6.13}$$

Substituting this expression into equation (6.11) one obtains

$$\begin{aligned} \lambda \cdot p^*(y) &= \int_0^y \left(p^*(u) - \frac{1}{\lambda} \cdot \int_y^1 p^*(v)dv\right) du + \int_y^1 p^*(u)du \\ &= \int_0^1 p^*(u)du - \frac{1}{\lambda} \cdot \int_0^y \int_y^1 p^*(v)dvdu \\ &= \int_0^1 p^*(u)du - \frac{1}{\lambda} \cdot y \cdot \int_y^1 p^*(v)dv. \end{aligned}$$

That is, $p^*(y)$ satisfies the integral equation

$$\lambda^2 \cdot p^*(y) = \lambda \cdot \int_0^1 p^*(u)du - y \cdot \int_y^1 p^*(v)dv. \tag{6.14}$$

Differentiating this equation twice we obtain

$$\lambda^2 \cdot p^{*\prime}(y) = y \cdot p^*(y) - \int_y^1 p^*(v)dv, \tag{6.15}$$

$$\lambda^2 \cdot p^{*\prime\prime}(y) = y \cdot p^{*\prime}(y) + 2 \cdot p^*(y). \tag{6.16}$$

The solution of this differential equation is given by

$$p^*(y) = C_1 \cdot y \cdot \exp\left(\frac{y^2}{2\lambda^2}\right) + C_2 \cdot \left[2 \cdot \lambda + \sqrt{2\pi} \cdot y \cdot \exp\left(\frac{y^2}{2\lambda^2}\right) \cdot \text{erf}\left(\frac{y}{\sqrt{2}\lambda}\right)\right]. \tag{6.17}$$

Setting $y = 0$ in equations (6.14) and (6.15) we obtain $\lambda \cdot p^*(0) = \int_0^1 p^*(u)du = -\lambda^2 \cdot p'(0)$. Inserting this condition into the solution of the differential equation (6.17) we obtain $C_1 = -2 \cdot C_2$. Moreover setting $C_2 = 1$ we obtain equation (6.10). □

Lemma 6.4. *The eigenfunctions of the operator T^* associated with the eigenvalue 0 has the form*

$$\psi(x, y) = \begin{cases} 0 & \text{if } 0 \le x \le y \le 1, \\ q^*(x, y) & \text{if } 0 \le y \le x \le 1 \end{cases}$$

where $q^*(x, y)$ satisfies $\int_0^y q^*(y, u)du = 0$.

Proposition 6.5. *For a non-zero eigenvalue λ with eigenvector φ and adjoint eigenvector ψ, we have*

$$\begin{aligned} (\mathbf{1}, \varphi) &= \lambda^2, \\ (\psi, \mathbf{1}) &= 2 \cdot \lambda^3, \\ (\psi, \varphi) &= 2 \cdot \lambda^2 \cdot \exp(-1/(2\lambda^2)). \end{aligned}$$

In particular,

$$\frac{(\psi, \mathbf{1})\,(\mathbf{1}, \varphi)}{(\psi, \varphi)} = \lambda^3 \cdot \exp(1/(2\lambda^2)).$$

Proof. In the calculations that follows we will use the relations $\operatorname{erf}(1/(\sqrt{2}\lambda)) = \sqrt{2/\pi}$, $\int_0^1 q(x)dx = \lambda$ and $\int_0^1 p^*(y)dy = 2 \cdot \lambda^2$.

First the inner product between the eigenfunction and the constant function $\mathbf{1}$:

$$\begin{aligned} (\mathbf{1}, \varphi) &= \int_{[0,1]^2} q(x)dxdy - \frac{1}{\lambda} \int_{0 \le x \le y \le 1} \int_x^y q(t)dtdxdy \\ &= \int_0^1 q(x)dx - \frac{1}{\lambda} \int_0^1 t \cdot (1 - t) \cdot q(t)dt \\ &= \lambda - \lambda + \frac{\sqrt{\pi}}{\sqrt{2}} \cdot \lambda^2 \cdot \operatorname{erf}(1/(\sqrt{2}\lambda)) \\ &= \lambda^2. \end{aligned}$$

Second, the inner product between the adjoint eigenfunction and the constant function $\mathbf{1}$. We have

$$\begin{aligned} (\psi, \mathbf{1}) &= \int_{[0,1]^2} p^*(y)dxdy - \frac{1}{\lambda} \int_{0 \le y \le x \le 1} \int_x^1 p^*(u)dudxdy \\ &= \int_0^1 p^*(y)dy - \frac{1}{\lambda} \int_0^1 \frac{u^2}{2} \cdot p^*(u)du \\ &= \int_0^1 \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot p^*(y)dy \\ &= \int_0^1 \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot \left(-2 \cdot y \cdot \exp\left(\frac{y^2}{2\lambda^2}\right) + 2 \cdot \lambda\right) dy \\ &+ \int_0^1 \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot \sqrt{2\pi} \cdot y \cdot \exp\left(\frac{y^2}{2\lambda^2}\right) \cdot \operatorname{erf}\left(\frac{y}{\sqrt{2}\lambda}\right) dy. \end{aligned}$$

The first integral is given by $I_1 = (\lambda - 2\lambda^2 - 2\lambda^3) \cdot \exp(1/(2\lambda^2)) - 1/3 + 2(\lambda + \lambda^2 + \lambda^3)$. The second integral we solve by integration by parts letting $f' = \left(1 - \frac{y^2}{2 \cdot \lambda}\right) \cdot \sqrt{2\pi} \cdot y \cdot \exp\left(\frac{y^2}{2\lambda^2}\right)$ and $g = \operatorname{erf}\left(\frac{y}{\sqrt{2}\lambda}\right)$. Then $\int_0^1 f' g dy = [fg]_0^1 - \int_0^1 f g' dy$ is given by:

$$\begin{aligned} I_2 &= \left[\frac{\sqrt{\pi}}{\sqrt{2}} \cdot \lambda \cdot (2\lambda + 2\lambda^2 - y^2) \cdot \exp(y^2/(2\lambda^2)) \cdot \operatorname{erf}(y/(\sqrt{2}\lambda)) \right]_0^1 \\ &- \int_0^1 (2\lambda + 2\lambda^2 - y^2) dy. \end{aligned}$$

Combining all the terms in the sum $I_1 + I_2$ we obtain $2 \cdot \lambda^3$. The third inner product is given by

$$\begin{aligned} (\psi, \varphi) &= \int_{[0,1]^2} q(x) \cdot p^*(y) dx dy \\ &- \frac{1}{\lambda} \int_{0 \le x \le y \le 1} \int_x^y q(t) dt \cdot p^*(y) dx dy - \frac{1}{\lambda} \int_{0 \le y \le x \le 1} q(x) \cdot \int_x^1 p^*(u) du dx dy \\ &= \left(\int_0^1 q(x) dx \right) \cdot \left(\int_0^1 p^*(y) dy \right) - \frac{2}{\lambda} \int_{0 \le t \le y \le 1} t \cdot q(t) \cdot p^*(y) dt dy \\ &= 2 \cdot \lambda^3 - 2\lambda \cdot \int_0^1 \left(1 - \exp(-y^2/(2\lambda^2))\right) \cdot p^*(y) dy \\ &= -2 \cdot \lambda^3 + 2\lambda \cdot \int_0^1 \exp(-y^2/(2\lambda^2)) \cdot p^*(y) dy \\ &= -2 \cdot \lambda^3 + 4\lambda \cdot \int_0^1 \left(-y + \exp(-y^2/(2\lambda^2)) + \sqrt{\pi}/\sqrt{2} \cdot y \cdot \operatorname{erf}(y/(\sqrt{2}\lambda)) \right) dy \\ &= 2 \cdot \lambda^2 \cdot \exp(-1/(2\lambda^2)). \end{aligned}$$

□

6.2. **Asymptotics.** We know that the largest root λ_0 of the eigenvalue equation (6.3) is real and positive, since the associated operator T is positivity improving. However, to say a bit more about the eigenvalues consider the related equation $\operatorname{erf}(z) = \sqrt{2/\pi}$.

Since the error function is an increasing function on the real axis, the equation $\operatorname{erf}(z) = \sqrt{2/\pi}$ has a unique real root. The error function is an odd function hence we know by the strong version of the little Picard theorem that the equation $\operatorname{erf}(z) = \sqrt{2/\pi}$ has infinitely many roots. Moreover, the complex roots appear in conjugate pairs. To summarize this discussion we have: The eigenvalue equation has a unique real root which is positive and is the largest root. The remaining infinite many roots are all complex and appear in conjugate pairs.

Numerically, we can approximate the roots of the eigenvalue equation (6.3). The unique real root is $\lambda_0 = 0.7839769312\ldots$. The next four largest roots are:

$$\begin{aligned} \lambda_{1,2} &= 0.2141426360\ldots \pm 0.2085807022\ldots \cdot i \\ \lambda_{3,4} &= -0.1677323922\ldots \pm 0.2418627350\ldots \cdot i \end{aligned}$$

¿From Proposition 6.5 we conclude:

Theorem 6.6. *The number of* 213*-avoiding permutations satisfies*

$$\frac{\alpha_n(213)}{n!} = \exp\left(\frac{1}{2\lambda_0^2}\right) \cdot \lambda_0^{n+1} + \mathcal{O}\left(|\lambda_1|^n\right)$$

where λ_0 *is the unique real root of the equation* $\operatorname{erf}\left(1/(\lambda\sqrt{2})\right) = \sqrt{2/\pi}$ *and* λ_1 *is the next largest root.*

7. Concluding remarks

It is straightforward to design a Viennot "pyramid" to compute the number α_n of S-avoiding permutations. For the original Viennot triangle, see [20, 21]. Let the entry $\alpha_n^{i_1,\ldots,i_m}$ of the pyramid be the number of permutations in the symmetric group on n elements, avoiding the set S and ending with the m entries $i_1, \ldots, i_m$. Then the entry $\alpha_n^{i_1,\ldots,i_m}$ is a sum of entries of the form $\alpha_{n-1}^{j,i_1,\ldots,i_{m-1}}$. This sum being a discrete analogue of the operator T. How far does this analogue between the discrete model and the continuous one go? Does the function $f_n = T^{n-m}(\mathbf{1})$ approximate the n-th level of the pyramid? More exactly, how well does the integer $\alpha_n^{i_1,\ldots,i_m}$ compare with $n! \cdot f_n(i_1/n, \ldots, i_m/n)$?

In the case of descent pattern avoidance, can one prove that T restricted to the invariant subspace V is compact? We have done so in the case of 123-avoiding permutations.

Consider the graph $G_\emptyset$ of overlapping permutations on the vertex set $\mathfrak{S}_m$. What is the smallest number of edges one has to remove in order to make the graph not strongly connected? Clearly one can remove m edges disconnecting the vertex $12\cdots m$. Is m the right answer? This would suggest that one can remove $m-1$ edges without making the directed graph disconnected.

A more general enumeration problem is as follows. For a function w on $\mathfrak{S}_{m+1}$ define the weight of a permutation $\pi = (\pi_1\pi_2\cdots\pi_n)$ by the product

$$\operatorname{wt}(\pi) = \prod_{k=1}^{n-m} w(\Pi(\pi_k, \ldots, \pi_{k+m})).$$

Now what can be said about the values and asymptotics of the sum

$$\alpha_n(w) = \sum_{\pi \in \mathfrak{S}_n} \operatorname{wt}(\pi)$$

as n tends to infinity. For instance, if w is a positive function we know by Kreĭn and Rutman that

$$\alpha_n(w) \sim c \cdot n! \cdot \rho^n.$$

In this paper our object is to understand consecutive patterns avoidance. Generalized pattern avoidance was introduced by Babson and Steingrímsson [1]. Is there an analytic approach to obtain asymptotics for these classes of permutations? Lastly, it would be daring to ask for an analytic proof of the former Stanley-Wilf conjecture, recently proved in [14].

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