

The Spectral Geometry of Geometrically Finite Hyperbolic Manifolds

Peter Perry

Dedicated to Barry Simon on his sixtieth birthday

ABSTRACT. Selberg's trace formula for a compact Riemann surface gives a precise, quantitative relationship between periods of closed geodesics and eigenvalues of the Laplacian. In this paper we discuss analogues of Selberg's trace formula for hyperbolic manifolds of infinite volume and the relationships they reveal between periods of closed geodesics and scattering resonances. We also discuss results for scattering theory on asymptotically hyperbolic manifolds which shed further light on this relationship and reveal interesting connections with conformal geometry.

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1. Introduction

For those of us who came of age in the 1980's, "geometric scattering theory" evokes the geometric approach to quantum-mechanical scattering which began with the pioneering work of Deift-Simon [34], Enss [40], [42], [41], Mourre [115], Sigal-Soffer [154], and others, reviewed in this volume in the survey of Christian Gérard [54]. Here I would like to survey recent developments in "geometric scattering

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theory” in Melrose’s [112] sense: the application of ideas and techniques of scattering theory to the study of the Laplacian on a complete, non-compact Riemannian manifold. For complete Riemannian manifolds with “simple geometry at infinity,” the study of the Laplace operator naturally leads to questions of spectrum, analytic continuation of the resolvent, and scattering resonances familiar to workers in Schrödinger operators

Manifolds of constant negative curvature and infinite volume are a class of manifolds with “simple geometry at infinity” which are particularly appealing in Mathematical Physics. On the one hand, the geodesic flow is ergodic, and there are infinitely many closed orbits; on the other, the Laplacian has at most finitely many eigenvalues and infinitely many resonances, and so provides a model of chaotic quantum scattering. The relationship between geodesic flow and the resonances can be analyzed in depth using ideas originally developed by Selberg [151] with applications to analytic number theory in mind.

The simplest example of the class of manifolds to be considered here is the hyperbolic cylinder X which arises as the quotient of the Poincaré upper half-plane by an abelian discrete group of isometries. Recall that the Poincaré upper-half plane is¹

$$\mathbb{H}^2 = \{(y, x) : y \in \mathbb{R}, x > 0\}$$

with the hyperbolic metric

$$ds^2 = x^{-2}(dx^2 + dy^2)$$

For any $\ell > 0$, the dilation

$$(y, x) \mapsto (e^\ell y, e^\ell x)$$

is an isometry of \mathbb{H}^2 and generates an abelian discrete group Γ . The quotient $X = \Gamma \backslash \mathbb{H}^2$ is a topological cylinder $\mathbb{R}_t \times S_\theta^1$ with the hyperbolic metric

$$ds^2 = dt^2 + \ell^2 \cosh^2 t d\theta^2$$

This manifold has closed geodesic of length ℓ at $t = 0$, and the ‘boundary at infinity’ is the union of two circles. Let Δ_X be the positive Laplacian on X . By Fourier analysis in the θ variable, it is easy to see that the operator $\Delta_X - 1/4$ is a direct sum over $m \in \mathbb{Z}$ of one-dimensional Schrödinger operators

$$L_m = -\frac{d^2}{dt^2} + \left[\left(\frac{2\pi m}{\ell} \right)^2 + \frac{1}{4} \right] \operatorname{sech}^2 t.$$

The spectral resolution of this operator can be explicitly calculated in terms of Legendre functions (see, for example, [74] and [30]). From this calculation it is clear that Δ_X has purely continuous spectrum in $[1/4, \infty)$. To study the resolvent $(\Delta_X - z)^{-1}$, we write the spectral parameter z as $s(1 - s)$ for $\operatorname{Re}(s) > 1/2$, corresponding to cut plane $\mathbb{C} \setminus [1/4, \infty)$. The resolvent has a meromorphic continuation to the complex s -plane with an infinite lattice of scattering resonances at the points

$$s = -k + \frac{2\pi im}{\ell}$$

¹The notation (y, x) for an ordered pair with $x > 0$ is motivated by a notational convention in geometric scattering theory in which x is a smooth positive function which vanishes to first order at the boundary at infinity; see section 6 below!

for $k = 0, 1, 2, \dots$ and $m \in \mathbb{Z}$. These should be thought of as quasi-bound states associated to the closed geodesic ℓ whose exponential rate of decay is determined by $\operatorname{Re}(s)$. They are exactly the zeros of the trivial ‘zeta function’

$$Z_X(s) = \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right)$$

The resonances and the resonance eigenfunctions determine a ‘resonance wave expansion’ for solutions of the non-Euclidean wave equation $u_{tt} + (\Delta_X - 1/4)u = 0$: see [30].

We will see that Selberg’s zeta function can be defined for a large class of hyperbolic manifolds, and connects the set of periods of closed orbits with the resonances of the Laplacian in much the same way as in the simple example above. Scattering theory plays an essential role in developing the trace formula and showing that the zeta function is a quotient of entire functions of bounded order. The theory of entire functions may then be used to derive the trace formula. Many of the well-known consequences of Selberg’s trace formula for compact surfaces, such as the asymptotics of the counting function for closed geodesics and the equidistribution of geodesics in homology classes, also hold in the infinite-volume setting. On the other hand, the distribution of resonances reflects features of the classical orbits such as the Hausdorff dimension of the ‘trapped set.’

In what follows, I will first review hyperbolic geometry, give some important examples of hyperbolic manifolds, and recall basic results about the spectrum of the Laplace operator. I will then discuss spectral and scattering theory for hyperbolic surfaces of infinite area, where results are most complete. In higher dimensions, I restrict attention to the class of convex co-compact hyperbolic manifolds where many of the same connections between quantum resonances and classical orbits have been established. Finally, I discuss recent work on scattering for asymptotically hyperbolic manifolds—a geometrically natural generalization of hyperbolic manifolds—where analogues of the trace formula hold and surprising connections with conformal geometry come to light.

This article follows the trajectory of my own work and the related research with which I am most familiar. For the relationship of questions discussed here to larger questions in the theory of scattering resonances, see, for example, the surveys of Zworski [161], [162], [169]. For a guide to the extensive literature on ‘quantum chaos’ for compact and finite-volume hyperbolic surfaces which we have not touched on in this paper, see, for example, Sarnak’s lectures [149] and [150]. Another important line of research not discussed in depth here is the work of Bunke and Olbrich on scattering theory for geometrically finite hyperbolic manifolds. Their approach to scattering theory yielded a proof of Patterson’s conjecture [128] characterizing the singularities of the zeta function in cohomological terms: see the monograph of Andreas Juhl [88], Bunke and Olbrich’s paper [23], and references contained therein. Melrose’s Stanford lectures [112] provide a broader perspective on “geometric scattering theory” as discussed here. For a survey of ‘counterexamples’ consisting of non-isometric pairs and families of non-compact manifolds with the same resonances, see [57]. Finally, Alice Chang’s lectures [28] provide the context in conformal geometry and geometric analysis in which Graham and Zworski’s results [61], discussed in section 6, may be more fully appreciated.

Acknowledgement. This article is tendered as a small expression of my appreciation for the wonderful job Barry Simon did as a thesis advisor, for the productive years that I spent at Caltech under his sponsorship, and for the enjoyable and rewarding career that has followed. On this occasion it should be duly noted that, although I moved outside the mainstream of Schrödinger operator theory, I could not escape the reach of Barry’s scholarship: see [33]!

I am also grateful to the mathematicians with whom I have had the opportunity to collaborate on some of the work discussed here, including David Borthwick, the late Robert Brooks, Richard Froese, Carolyn Gordon, Peter Hislop, Chris Judge, Jeffrey McGowan, Ruth Gornet, S. J. Patterson, Dorothee Schueth, and Floyd Williams. I thank David Borthwick for sharing with me a first draft of his forthcoming book on scattering theory for Riemann surfaces. Finally, I am grateful to the referee and to Laurent Guillopé for a careful reading of the manuscript and many suggestions that improved the finished product.

2. Geometrically Finite Hyperbolic Manifolds

Hyperbolic manifolds arise as quotients of real hyperbolic space by a discrete group Γ of isometries. Let us first recall some basic notions of hyperbolic geometry and discrete groups of hyperbolic isometries. For a complete and systematic development of hyperbolic geometry we refer the reader to the monograph of Ratcliffe [144].

Real $(n + 1)$ -dimensional hyperbolic space \mathbb{H}^{n+1} may be realized either as the open unit ball

$$\{z \in \mathbb{R}^{n+1} : |z| < 1\}$$

with metric

$$ds^2 = \frac{4 dz^2}{1 - |z|^2}$$

or as the upper half-space

$$\mathbb{R}_+^{n+1} = \{(y, x) : y \in \mathbb{R}^n, x > 0\}$$

and metric

$$g = x^{-2}(dx^2 + dy^2).$$

Isometries of \mathbb{H}^{n+1} are Möbius transformations leaving the unit ball (resp. the upper half-space) invariant. The isometries of \mathbb{H}^{n+1} act transitively, i.e., any point in \mathbb{H}^{n+1} can be mapped to any other point of \mathbb{H}^{n+1} by some isometry. These isometries may be divided into three types. The first type is an elliptic motion or non-Euclidean rotation conjugate in the ball model to a Euclidean rotation about the origin. The second type is parabolic motion conjugate to a mapping of the form

$$(y, x) \mapsto (Ay + b, x)$$

in the upper half-space model (where A is an orthogonal matrix and b a fixed vector in \mathbb{R}^n). The third type is a loxodromic motion conjugate in the upper half-space model to a mapping of the form

$$(2.1) \quad (y, x) \mapsto e^{\ell(\gamma)}(A_\gamma y, x)$$

where $\ell_\gamma > 0$ and $A_\gamma \in O(n)$. We will denote by $\alpha_1(\gamma), \dots, \alpha_n(\gamma)$ the eigenvalues of A_γ . The group of isometries of \mathbb{H}^{n+1} is isomorphic to the group of conformal transformations on the ‘boundary at infinity.’

Geodesics in the upper half-space model are straight lines parallel to the x -axis, and semicircles intersecting the boundary $x = 0$ normally. Any geodesic is determined by its endpoints at the boundary. Hemispheres intersecting the boundary normally are geodesic hyperplanes which divide \mathbb{H}^{n+1} into two isometric half-spaces. Let $d(\cdot, \cdot)$ denote hyperbolic distance. We say that two geodesics $\gamma_1(t)$ and $\gamma_2(t)$ are equivalent if $d(\gamma_1(t), \gamma_2(t))$ remains bounded as $t \rightarrow \infty$. It is not difficult to see that equivalence classes of geodesics are labelled by boundary points in S^n (in the ball model), so that S^n is the ‘geometric infinity’ of \mathbb{H}^{n+1} . In dynamical terms, S^n parameterizes the possible set of directions for classical paths moving to infinity.

If Γ is an orientation-preserving discrete group of isometries of \mathbb{H}^{n+1} , we say that Γ is *geometrically finite* if there is a finite-sided fundamental domain for Γ in \mathbb{H}^{n+1} . We will always assume that Γ is torsion-free (i.e., has no elements of finite order), so that $X = \Gamma \backslash \mathbb{H}^{n+1}$ is a Riemannian manifold with the Riemannian metric induced from the Poincaré metric on \mathbb{H}^{n+1} . The *limit set* of Γ , denoted $\Lambda(\Gamma)$, is most easily described in the unit ball model: it is the subset of S^n consisting of accumulation points of Γ -orbits in the Euclidean topology of \mathbb{B}^{n+1} . It is not difficult to see that if z_0 is any fixed point in \mathbb{H}^{n+1} , the set of accumulation points of the orbit $\{\gamma(z_0) : \gamma \in \Gamma\}$ is independent of the choice of z_0 and so is equal to $\Lambda(\Gamma)$. The limit set is a closed subset of S^n , and its complement, denoted $\Omega(\Gamma)$, is called the *domain of discontinuity* of Γ acting on S^n . The (hyperbolic) convex hull of the limit set is the union of all geodesics in \mathbb{H}^{n+1} with endpoints in the limit set.

The set of closed geodesics of Γ are in one-to-one correspondence with the hyperbolic conjugacy classes of Γ . A hyperbolic element in Γ is called *primitive* if it is not the power of another element in Γ ; analogously, a closed geodesic is called *primitive* if it is not the power (in the sense of composition of paths) of another closed geodesic. The primitive closed geodesics of X are in one-to-one correspondence with the primitive hyperbolic conjugacy classes of Γ .

The group Γ is called *convex co-compact* if X has infinite volume but any fundamental domain for the action of Γ on \mathbb{H}^{n+1} intersects the convex hull of the limit set in a compact convex subset of \mathbb{H}^{n+1} . If Γ is convex co-compact, it contains no parabolic elements and the quotient $X = \Gamma \backslash \mathbb{H}^{n+1}$ is the interior of a compact manifold with boundary (see Example 2.2) below.

The *exponent of convergence* of Γ is the infimum over those real s for which the Poincaré series

$$\sum_{\gamma \in \Gamma} \exp[-sd(x, \gamma(x))]$$

converges; here $d(\cdot, \cdot)$ is hyperbolic distance. The exponent δ is independent of $x \in \mathbb{H}^{n+1}$ by the triangle inequality and the fact that γ is an isometry.

We now consider several important examples of hyperbolic manifolds.

EXAMPLE 2.1. (Hyperbolic surfaces) Suppose that Γ is a geometrically finite discrete group of hyperbolic isometries acting on \mathbb{H}^2 . The quotient $X = \Gamma \backslash \mathbb{H}^2$ has the decomposition

$$(2.2) \quad X = \widehat{X} \sqcup (\sqcup_{i=1}^{n_C} C_i) \sqcup (\sqcup_{j=1}^{n_F} F_j)$$

where \widehat{X} is a compact manifold with boundary, and the C_i and F_j are ‘cusp’ and ‘funnel’ ends described as follows. The C_i are cuspidal ends isometric to $S^1_\theta \times [0, \infty)_t$ with the metric

$$(2.3) \quad ds^2 = dt^2 + h_i^2 e^{-2t} d\theta^2$$

(so that, metrically, the “infinity” of the cusp is a single point), where h_i is the length of the horocycle joining C_i to the compact manifold with boundary X . The F_j are funnel ends isometric to $S_\theta^1 \times [0, \infty)_t$ with the metric

$$(2.4) \quad ds^2 = dt^2 + \ell_j^2 \cosh^2(t) d\theta^2$$

(so that, metrically, the “infinity” of the cusp is a circle). Here ℓ_j is the hyperbolic length of the geodesic circle where F_j is joined to \widehat{X} . All of the closed geodesics of X lie in \widehat{X} . The integers n_C and n_F count the respective number of cusp and funnel ends of X , and we assume that at least one of n_F and n_C is nonzero. The boundary of \widehat{X} consists of n_F simple closed geodesics where the funnels are attached, and n_C circles where the cusps are attached. The boundary at infinity of X thus consists of n_C ideal points and n_F circles. If $n_F = 0$ then X has finite hyperbolic volume but if $n_F > 0$, then X has infinite hyperbolic volume. In what follows we will always suppose that $n_F > 0$. Topologically, X is a surface of genus h (the number of holes in \widehat{X}) with n_C punctures and n_F discs removed, and the Euler characteristic of X is given by

$$(2.5) \quad \chi(X) = 2 - 2h - 2n_F - 2n_C$$

EXAMPLE 2.2. (Convex co-compact hyperbolic manifolds) Suppose that Γ is a convex co-compact discrete group of hyperbolic isometries acting on \mathbb{H}^{n+1} . By definition, the hyperbolic convex hull of the limit set intersects any fundamental domain for the action of Γ on \mathbb{H}^{n+1} in a convex compact set, the Nielsen region for Γ . This descends to $X = \Gamma \backslash \mathbb{H}^{n+1}$ as a compact convex set, \widehat{X} , called the convex core of X (in two dimensions, X has smooth boundary, but this need not be the case in higher dimensions). The Euler characteristic of X is that of the compact manifold with boundary \widehat{X} . Thus X is the union of \widehat{X} and finitely many ‘ends’ diffeomorphic to $(0, 1) \times S$ for a compact manifold S without boundary. These ends are analogous to the funnels of Example 2.1 but need not carry a simple warped product metric if $\dim(X) > 2$. Thus, in general, there is no simple model Laplacian for the ends of a convex co-compact hyperbolic manifold in dimension three or higher. There is a natural geometric compactification of X to a manifold with boundary. Let $M = \Gamma \backslash \Omega(\Gamma)$ (recall that $\Omega(\Gamma)$ is the domain of discontinuity of Γ acting on the boundary at infinity of \mathbb{H}^{n+1}); this manifold carries a natural conformal structure since the transformations of Γ act conformally on $\Omega(\Gamma)$. Then $\overline{X} = X \cup M$ is a compact manifold with boundary. The smooth manifold M parameterizes the directions in which a geodesic may escape to infinity just as the manifold S^n parameterizes these directions for geodesics in the space \mathbb{H}^{n+1} . In example 2.1, a surface with $n_C = 0$ is convex co-compact, and M is a disjoint union of n_F circles.

EXAMPLE 2.3. (Classical Schottky groups) For simplicity we will describe classical Schottky groups acting on \mathbb{H}^3 , following [21]; see also section 2 of [73] for an introduction to Schottky groups and further references. Suppose that

$$\{C_j : 1 \leq j \leq 2g\}$$

is a collection of disjoint circles on the Riemann sphere S^2 (viewed as the boundary of the unit ball in \mathbb{R}^3). For each i , $1 \leq i \leq g$, let γ_i be a Möbius transformation with $\gamma_i(C_{2i-1}) = C_{2i}$ and taking the interior of C_{2i-1} to the exterior of C_{2i} . Each circle C_i bounds a hyperbolic half space H_i , and each γ_i extends to a hyperbolic

isometry with $\gamma_i(\partial H_{2i-1}) = \partial H_{2i}$ and $\gamma_i(H_{2i-1}) = \mathbb{H}^3 \setminus H_{2i}$. If Γ is the free discrete group generated by the γ_i , the group Γ is called a Schottky group. A fundamental domain for its action on \mathbb{H}^3 is $\mathbb{H}^3 - \cup_{i=1}^{2g} H_j$. It is not difficult to see that the quotient manifold $X = \Gamma \setminus \mathbb{H}^3$ is handlebody of genus g whose boundary at infinity is a Riemann surface of genus g . Schottky groups have the property that geodesic flow on the quotient X can be coded as symbolic dynamics, and the dynamical zeta function for geodesic flow can be studied using the Ruelle theory of transfer operators: see [73] and see our discussion in section 5.3 below.

3. Spectral Theory for the Laplacian

Lax and Phillips [94], [95], [96] studied the spectral theory of the Laplace operator on geometrically finite hyperbolic manifolds in the framework of Lax-Phillips scattering theory [97].

THEOREM 3.1. [94], [95] *Suppose that Γ is a geometrically finite discrete group of isometries of \mathbb{H}^{n+1} so that $X = \Gamma \setminus \mathbb{H}^{n+1}$ has infinite volume. Then Δ_X has at most finitely many eigenvalues in the interval $[0, n^2/4)$ and purely absolutely continuous spectrum with no embedded eigenvalues in $[n^2/4, \infty)$.*

The absolute continuity follows from the translation representation established in [95], which provides much more detailed information about the behavior of solutions to the non-Euclidean wave equation than is discussed here. It follows that the resolvent $(\Delta_X - z)^{-1}$ is meromorphic in the cut plane $[n^2/4, \infty)$ with at most finitely many poles due to eigenvalues. It is convenient to set $z = s(n - s)$ and define

$$R_X(s) = (\Delta_X - s(n - s))^{-1}$$

initially in the half-plane $\text{Re}(s) > n/2$, corresponding to the cut plane $[n^2/4, \infty)$. The continuous spectrum corresponds to the ‘critical line’ $\text{Re}(s) = n/2$. In the following sections we will consider meromorphic continuation of the resolvent and the geometry of the resolvent resonances.

The Patterson-Sullivan Theorem [125], [159] relates the exponent of convergence δ for the discrete group Γ to the spectrum of the Laplace operator.

THEOREM 3.2. [125], [159] *Suppose that Γ is a geometrically finite discrete group of hyperbolic isometries acting on \mathbb{H}^{n+1} with exponent of convergence δ . Then the limit set $\Lambda(\Gamma)$ has Hausdorff dimension δ . Moreover, if $\delta > n/2$, the lowest eigenvalue of the Laplacian is $\delta(n - \delta)$, and the positive eigenfunction can be recovered from a Poisson integral over the Hausdorff measure on the limit set.*

The Theorem is due to Patterson if $\dim(X) = 2$ and to Sullivan for general n . It implies that the resolvent is pole-free for $\text{Re}(s) > \delta$. If $\delta \leq n/2$ it is still the case that the first pole of the meromorphically continued resolvent is given by δ unless $\delta - n/2$ is an integer; see [127].

We close this section with a brief survey of related work on the spectral and scattering theory of hyperbolic manifolds of infinite volume. The spectral theory of the Laplacian on infinite volume, hyperbolic surfaces (i.e., $n_F > 0$ in Example 2.1) was first studied by Elstrodt [38], Fay [45], Guillopé [70], and Patterson [124]; these surfaces, as quotients of \mathbb{H}^2 by a geometrically finite discrete group, are included in the class of geometrically finite hyperbolic manifolds (in any dimension)

studied by Lax and Phillips in [95], [96]. If $n_C = 0$ but $n_F > 0$ then X is a convex co-compact hyperbolic surface; we discuss this case and its higher-dimensional analogues in greater detail in section 5. The case $n_F = 0$ but $n_C > 0$ includes the quotient of \mathbb{H}^2 by arithmetic subgroups of $PSL(2, \mathbb{R})$ and has been the subject of intensive investigation by many mathematicians; among the many studies of spectral and scattering theory in this setting, we mention only the pioneering work of Faddeev [44] and the 1976 study of Lax and Phillips [93]. Colin de Verdière [29] studied a counting problem for geodesics on hyperbolic surfaces related to the counting problems discussed in section 4.3 which uses the spectral theory of the Laplacian. Mandouvalos [100] studied three-dimensional hyperbolic manifolds using techniques of analytic number theory, while Agmon [1], [2] and Perry [131], [132] applied Schrödinger-operator techniques to study spectral and scattering theory for convex co-compact hyperbolic manifolds. Mazzeo and Melrose [101] used very different techniques to study the Laplace operator on a large class of manifolds, the asymptotically hyperbolic manifolds, that include convex co-compact hyperbolic manifolds. Their techniques have been applied to a number of problems on asymptotically hyperbolic manifolds and we will return to their work later. Froese, Hislop, and Perry [51],[52], building on earlier work of Froese and Hislop [49], used Mourre’s commutator method and Schrödinger operator techniques to study the spectral and scattering theory for general geometrically finite hyperbolic manifolds in three dimensions, including those with so-called non-maximal rank cusps. These authors prove that the scattering operator, and hence the resolvent, admit a meromorphic continuation to the complex plane. Recently, Mourre’s commutator technique has been used by Bouclet [18] to prove high-energy resolvent estimates for asymptotically hyperbolic manifolds. Cordoso and Vodev [31] proved optimal high-energy estimates on the resolvent for a class of manifolds including hyperbolic manifolds with cusps. Bunke and Olbrich [24] studied spectral theory of the Laplacian and analytic continuation of the scattering operator for general geometrically finite discrete groups (including those with irrational cusps) using techniques of representation theory. They also refined and proved Patterson’s conjecture [128] which relates the singularities of the zeta function to group cohomology of the group Γ and provides a unified approach to the ‘spectral’ and ‘topological’ zeros of the zeta function (see Theorem 5.3 below). Recently, Guillarmou [63] has proven analytic continuation of the resolvent for a class of geometrically finite discrete groups acting on \mathbb{H}^{n+1} , $n \geq 1$, which includes the class studied by Froese, Hislop and Perry but excludes manifolds with irrational cusps. The reader may consult, for example, the survey of Hislop [81] for further discussion and references to the literature up to 1993.

4. Spectral Geometry of Non-Compact Riemann Surfaces

In this section, we consider the spectral geometry of geometrically finite, non-compact hyperbolic surfaces of infinite volume, as described in Example 2.1 above. Because these surfaces have ends isometric to explicit model spaces, particularly sharp results have been proven on the distribution of resonances, the asymptotics of the length spectrum, and the inverse resonance problem. For a more comprehensive introduction to the spectral geometry of these surfaces, we refer the reader to David Borthwick’s forthcoming book [13].

There is a natural renormalized integral on X , the 0-integral of a smooth function on X with suitable asymptotic behavior at infinity, which will play an important role in some of the result that follow. To define it, recall the cusp and funnel coordinates (t, θ) from Example 2.1 and set, for $\varepsilon \in (0, 1)$:

$$X(\varepsilon) = \widehat{X} \sqcup (\sqcup_{i=1}^{n_C} C_i(\varepsilon)) \sqcup (\sqcup_{j=1}^{n_F} F_j(\varepsilon))$$

where

$$C_i(\varepsilon) = \{m \in C_i(\varepsilon) : t(m) \leq -\log \varepsilon\}$$

and

$$F_j(\varepsilon) = \{m \in F_j(\varepsilon) : t(m) \leq -\log \varepsilon\}$$

The compact submanifolds $X(\varepsilon)$ exhaust X . If $f \in C^\infty(X)$ and dg denotes Riemannian volume measure with respect to the hyperbolic metric, we set

$$(4.1) \quad \int_0 f dg = \text{FP}_{\varepsilon \downarrow 0} \int_{X(\varepsilon)} f dg$$

where FP denotes the Hadamard finite part (see Definition 1 of [77] and see Hörmander [84], section III.3.2, page 70 for discussion). The existence of the 0-integral depends on the asymptotic behavior of f in the cusp and funnel ends. To given sufficient conditions, put $\rho = e^{-t}$ where t is the appropriate cusp or funnel coordinate, and say that f is *polyhomogeneous* if in each end f admits an asymptotic expansion of the form

$$f(\rho, \theta) \sim \sum_{j=-N}^{\infty} \sum_{k=0}^{N_j} f_{j,k}(\theta) \rho^j \log^k \rho$$

as $\rho \downarrow 0$. Observe that, in each cusp or funnel, $\rho^2 dg$ is a smooth function of ρ and θ . It follows that $\int_{X(\varepsilon)} f dg$ has an asymptotic expansion of the form $\sum_{j,k} c_{j,k} \varepsilon^j (\log(1/\varepsilon))^k$ (where the j may be negative); the Hadamard finite part is the coefficient $c_{0,0}$. The zero-trace need not be invariantly defined (the definition given here depends on an explicit choice of coordinates in the cusp and funnel ends) but under some circumstances it leads to geometric invariants. A simple example is the 0-volume

$$(4.2) \quad 0\text{-vol}(X) = \int_0 dg = -2\pi\chi(X)$$

as may easily be seen from the Gauss-Bonnet Theorem for \widehat{X} and the explicit form of the cusp and funnel metrics. The 0-trace of an operator with smooth kernel is the 0-integral of the kernel of the diagonal.

4.1. Resolvent and Scattering Operator. For surfaces with $n_F > 0$, the Laplacian Δ_X has purely continuous spectrum of infinite multiplicity in $[1/4, \infty)$ and at most finitely many eigenvalues in $[0, 1/4)$. (see Theorem 3.1). It follows that the resolvent $(\Delta_X - z)^{-1}$ is meromorphic in the cut plane $\mathbb{C} \setminus [\frac{1}{4}, \infty)$. It will be convenient to make the quadratic transformation $z = s(1-s)$ of the spectral parameter so that the continuous spectrum lies on the line $\text{Re}(s) = \frac{1}{2}$ and the ‘physical region’ corresponds to the half-plane $\text{Re}(s) > 1/2$. We define

$$R_X(s) = (\Delta_X - s(1-s))^{-1}$$

as an operator on $L^2(X)$ for $\operatorname{Re}(s) > 1/2$ and $s(1-s)$ not an eigenvalue of Δ_X . Using the explicit form of model cusp and funnel resolvents, Guillopé and Zworski [74] prove:

THEOREM 4.1. [74] *The resolvent $R_X(s)$ extends to a meromorphic operator valued function from $\mathcal{C}_0^\infty(X)$ to $\mathcal{C}^\infty(X)$ with poles of finite rank.*

More precisely, $R_X(s)$ has isolated singularities and the Laurent expansion of $R_X(s)$ at each such singularity has a finite polar part whose coefficients are finite-rank, smoothing operators. *Resolvent resonances* are poles of the meromorphically continued resolvent.

We denote by \mathcal{R}_X the collection of poles of the resolvent (including those real s with $s > 1/2$ and $s(1-s)$ an eigenvalue of the Laplacian), and define the *multiplicity* of a singularity $\zeta \in \mathcal{R}_X$, denoted m_ζ , to be the rank of the residue

$$\frac{1}{2\pi i} \int_\gamma (2s-1)R_X(s) ds$$

where γ is a contour enclosing ζ and no other singularity of $R_X(s)$; note that if $\zeta > 1/2$ and $\zeta(1-\zeta)$ is an eigenvalue then m_ζ is the usual multiplicity. Let

$$N_X(r) = \#\{\zeta \in \mathcal{R}_X : |\zeta| \leq r\}.$$

Guillopé and Zworski [74] prove the upper bound $N_X(r) \leq C_1 r^2$. The upper bound implies that the Hadamard product

$$(4.3) \quad P_X(s) = \prod_{\zeta \in \mathcal{R}} (1-s/\zeta)^{m_\zeta} e^{m_\zeta(s/\zeta + s^2/2\zeta^2)}$$

converges to an entire function of s having order at most two. We will use the Hadamard product below to describe analytic properties of Selberg's zeta function on X . In [76], Guillopé and Zworski show that $N_X(r)$ also obeys a lower bound $N_X(r) \geq C_2 r^2$ (where C_2 is geometrically defined and may vanish for some X).

Scattering eigenfunctions for Δ_X are smooth solutions of the eigenvalue equation $(\Delta_X - s(1-s))u = 0$ for $\operatorname{Re}(s) = 1/2$. In the cusp ends, these can be chosen to take the asymptotic form

$$u(t, \theta) \sim a_i \rho^{1-s} + b_i \rho^s + \mathcal{O}(\rho^\infty)$$

where $\rho = e^{-t}$ (referring to the coordinates in (2.3)) while in the funnel ends

$$u(t, \theta) \sim \rho^{1-s} f_j(\theta) + \rho^s g_j(\theta) + \mathcal{O}(\rho)$$

where again $\rho = e^{-t}$ (referring to the coordinates in (2.4)). It turns out that for fixed s with $\operatorname{Re}(s) = 1/2$ and $s \neq 1/2$, a scattering eigenfunction is uniquely determined by the numbers $\{a_i\}_{i=1}^{n_C}$ and the functions $\{f_j\}_{j=1}^{n_F}$, i.e., by data in $\mathbb{C}^{n_C} \oplus (\oplus_{j=1}^{n_F} \mathcal{C}^\infty(S^1))$. Thus the map

$$S_X(s) : \mathbb{C}^{n_C} \oplus (\oplus_{j=1}^{n_F} \mathcal{C}^\infty(S^1)) \rightarrow \mathbb{C}^{n_C} \oplus (\oplus_{j=1}^{n_F} \mathcal{C}^\infty(S^1))$$

$$(\{a_i\}_{i=1}^{n_C}, \{f_j\}_{j=1}^{n_F}) \mapsto (\{b_i\}_{i=1}^{n_C}, \{g_j\}_{j=1}^{n_F})$$

is well-defined. From its definition it is clear that

$$S_X(s)S_X(n-s) = I.$$

This map is called the *absolute scattering operator* for X and relates ‘incoming’ to ‘outgoing’ asymptotic data for scattering eigenfunctions. The operator $S_X(s)$ is

a pseudodifferential operator (see section 5.1 below where the singularities of the scattering operator for convex co-compact manifolds are described).

The geometry of X suggests a natural ‘comparison dynamics’ for scattering. Each funnel end (imposing Dirichlet boundary conditions at $t = 0$) has an absolute scattering operator $S_{F_j}(s)$ acting on $\mathcal{C}^\infty(S^1)$ which may be explicitly calculated by separation of variables in the eigenvalue equation. If we define a ‘comparison’ scattering operator acting on $\mathbb{C}^{n_C} \oplus (\oplus_{j=1}^{n_F} \mathcal{C}^\infty(S^1))$ as

$$S_0(s) = \mathbf{1} \oplus (\oplus_{j=1}^{n_C} S_{F_j}(s))$$

(here $\mathbf{1}$ is the identity operator on \mathbb{C}^{n_C}), the relative scattering operator is given by

$$\mathcal{S}_X(s) = S_X(s)S_0(s)^{-1}$$

In [76] it is shown that, for $\operatorname{Re}(s) = 1/2$ and $s \neq 1/2$, the relative scattering operator takes the form $\mathcal{S}_X(s) = I + T_X(s)$ where $T_X(s)$ is a trace-class operator on the natural Hilbert space $\mathbb{C}^{n_C} \oplus (\oplus_{j=1}^{n_F} L^2(S^1))$. Thus the determinant

$$D_X(s) = \det(I + T_X(s))$$

is well-defined. The associated scattering phase is

$$\sigma_X(s) = \frac{i}{2\pi} \log \det(\mathcal{S}_X(s))$$

Guilopé and Zworski [76] prove that the scattering phase has Weyl-type asymptotics.

THEOREM 4.2. [76] *The relative scattering phase obeys the asymptotic formula*

$$\sigma_X(s) = \frac{1}{4\pi} \operatorname{vol}(X) |s|^2 - \frac{n_C}{\pi} |s| \log |s| + \mathcal{O}(|s|).$$

4.2. Resonances and Selberg’s Zeta Function. The set \mathcal{R}_X (including the multiplicities) may be regarded as the analogue of the eigenvalues of the Laplacian on a compact surface, and it is natural to ask how the resonances reflect the underlying geometry of X as encoded in the closed geodesics of X . For a compact manifold X , Selberg’s zeta function gives a precise relationship between the eigenvalues of X , the length spectrum of X , and the Euler characteristic $\chi(X)$ for X . The zeta function is defined as the Euler product

$$Z_X(s) = \prod_{\gamma \in \mathcal{P}_X} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right)$$

where the outer product ranges over primitive closed geodesics of X , and $\ell(\gamma)$ denotes the hyperbolic length of γ . This product converges for $\operatorname{Re}(s) > 1$ as a simple geometric argument² shows that the number of geodesics of length r grows at most like $\exp(r)$. It follows from Selberg’s trace formula (Selberg [151]; see also, for example McKean [107] for an exposition, but see Hejhal [80], Theorem 4.11,

²Fix an origin 0 in hyperbolic space and consider the lattice $\{\gamma(0) : \gamma \in \Gamma\}$. For each γ the points 0 and $\gamma(0)$ are separated by a geodesic hyperplane, so 0 is contained in an open half-space H_γ . An open fundamental domain for Γ is given by $\mathcal{F} = \cap_{\gamma \in \Gamma} H_\gamma$.

Since closed geodesics correspond to conjugacy classes of hyperbolic elements, counting lattice points in a metric ball of radius r provides an upper estimate on the counting function for closed geodesics. The volume of a geodesic ball about 0 grows as the exponential of the geodesic distance. If X (and hence \mathcal{F}) has finite volume, there can be at most $\mathcal{O}(\exp(r))$ lattice points within a metric ball of radius r , and if X has infinite volume, the same is true.

p. 72 and comments following for a precise formulation and a correction to [107]) that the zeta function has an analytic continuation to an entire function of order two having spectral zeros at those s with $\operatorname{Re}(s) \geq \frac{1}{2}$ and $s(1-s)$ is an eigenvalue of the Laplacian, a zero at $s = 0$ of multiplicity $1 - \chi(X)$, and topological zeros at the integers $s = -k$, $k = 1, 2, \dots$, with multiplicity $-(2k+1)\chi(X)$.

The zeta function remains well-defined when X is a hyperbolic surface of infinite volume, and it is natural to ask whether an analogous relationship holds. Guillopé [71], [72] showed that, for such X , the zeta function has a meromorphic continuation to the complex plane and zeros in the resonance \mathcal{R}_X . Borthwick, Judge, and Perry [15] determined the order and divisor of the zeta function.

THEOREM 4.3. [15] *Let $X = \Gamma \backslash \mathbb{H}^2$ where Γ is finitely generated, and suppose that $n_F > 0$. Selberg's zeta function extends to quotient of entire function of order at most two with the following singularities:*

- (1) *Spectral zeros (with multiplicity) at points $\zeta \in \mathcal{R}_X$*
- (2) *Topological zeros at $s = -k$, $k = 0, 1, 2, \dots$, of order $(2k+1)(-\chi(X))$*
- (3) *Topological poles at $s = \frac{1}{2} - k$, $k = 0, 1, 2, \dots$, of order n_C .*

For finite area surfaces ($n_F = 0$) the result is a well-known consequence of Selberg's trace formula. The proof of Theorem 4.3 relies heavily on the scattering theory developed by Guillopé and Zworski in [76] and an explicit formula relating the resolvent of the Laplacian Δ_X to the zeta function (see section 5.2 below for a formula valid when $n_C = 0$). The factorization

$$Z_X(s) = e^{q(s)} \Gamma(s-1/2)^{n_C} Z_\infty(s) P_X(s)$$

holds, where

$$Z_\infty(s) = \left[\frac{(2\pi)^s \Gamma_2(s)^2}{\Gamma(s)} \right]^{-\chi(X)}$$

($\Gamma_2(s)$ is Barnes' double Γ -function) and q is a polynomial of order at most two.

There is also a natural renormalized 'determinant of the Laplacian' (see [14] for the case $n_C = 0$ and [15] for the general case; compare [148] and [35] for compact surfaces, and [37] for non-compact surfaces of finite volume)

$$D_X(s) = \det(\Delta_X - s(1-s))$$

which obeys the formula

$$D_X(s) = e^{F s(s-1)+G} Z_X(s) / Z_\infty(s)$$

if $n_C = 0$ (here F and G are constants which depend on the renormalization; see [14]), and a similar formula with additional singular factors if $n_C > 0$ (see [15], Theorem 5.1). Here the determinant is defined, not by the usual ζ -function regularization, but rather by the formula

$$\left(\frac{1}{2s-1} \frac{d}{ds} \right)^2 \log D_X(s) = 0 - \operatorname{Tr} (R_X(s)^2)$$

which is motivated by the identity

$$\left(\frac{d}{d\lambda} \right)^2 \log \det (A - \lambda I) = \operatorname{Tr} \left[(A - \lambda I)^{-2} \right]$$

true for symmetric matrices.

Using the analytic properties and asymptotics of the zeta function, one can prove ([15], Corollary 1.2):

THEOREM 4.4. [15] *The resonance set \mathcal{R}_X determines the length spectrum of X , the Euler characteristic $\chi(X)$, and the number of cusps n_C . The length spectrum determines $\chi(X)$ and n_C up to finitely many possibilities. The length spectrum, $\chi(X)$, and n_C together determine the resonance set.*

The discrete group Γ has a realization as a finitely generated, torsion-free subgroup of $SL(2, \mathbb{R})$, and the lengths of closed geodesics of X correspond to traces of elements $\gamma \in \Gamma$. By combining Theorem 4.4 with Teichmüller theory we can prove a rigidity result analogous to those of McKean [107] for compact Riemann surfaces and Müller [116] for Riemann surfaces of finite area.

THEOREM 4.5. [15] *Let X be a complete, geometrically finite hyperbolic surface of infinite area. Then the length spectrum of X determines X up to finitely many possibilities. In particular, the resonance set determines X up to finitely many possibilities.*

There are examples of non-isometric Riemann surfaces of infinite area with the same scattering resonances (see Remark 2.15 in [76] and [20], both of which exploit Bérard's formulation [11] of Sunada's method [160]), so the conclusion of Theorem 4.5 is not trivial

It is natural to ask what happens when one considers perturbations of the hyperbolic metric. For compactly supported perturbations of the metric, all of the essential features of spectral theory—the meromorphic continuation of the resolvent, basic estimates on the counting function for closed geodesics, etc.—remain unchanged, as is proved by Guillopé and Zworski in [76]. Borthwick, Judge and Perry [14] considered relative determinants for metrics on infinite area surfaces X with $n_C = 0$ of the form $g = e^{2\varphi}\tau$ where τ is a hyperbolic metric and φ is a smooth, compactly supported function. An important role is played in this work by the uniformization theorem of Mazzeo and Taylor [105] for asymptotically hyperbolic metrics on surfaces. Using the relative determinant, they show first of all that the resonance set of the perturbed metric determines the Euler characteristic of X , and hence the diffeomorphism type of the surface, up to finitely many possibilities. They also show that the determinant obeys a Polyakov-type formula for conformal variations of the metric, and obtain an analogue of Osgood, Phillips, and Sarnak's celebrated result [122] that the set of metrics on a compact surface with given spectrum is precompact in a natural C^∞ topology on isometry classes.

A complementary formulation of the duality between closed geodesics and resonances has been given by Guillopé and Zworski [77], who prove a wave-trace formula analogous to the Duistermaat-Guillemin trace formula [36] for elliptic operators on compact manifolds. The set \mathcal{P}_X of primitive closed geodesics of least period $\ell(\gamma)$ is in one-to-one correspondence with the conjugacy classes of primitive elements in Γ . Let P_γ denote the Poincaré once-return map for geodesics viewed as closed orbits in the cotangent bundle T^*M . Guillopé and Zworski prove:

THEOREM 4.6. [77] *Let $X = \Gamma \backslash \mathbb{H}^2$, where Γ is finitely generated, having n_C cusps. The following formula holds in distribution sense for $t \in \mathbb{R}$:*

$$\begin{aligned} 0\text{-Tr} \left[\cos \left(t \sqrt{\Delta_X - \frac{1}{4}} \right) \right] &= -\frac{0\text{-vol}(X)}{8\pi} \frac{\cosh(t/2)}{\sinh^2(t/2)} \\ &+ \frac{n_C}{4} \coth(|t|/2) \\ &+ \left[n_C(\gamma_0 - \log 2) - \sum_{i=1}^{n_C} h_i \right] \delta(t) \\ &+ \frac{1}{2} \sum_{\gamma} \sum_{k=1}^{\infty} \frac{\ell(\gamma)}{|1 - P_{\gamma}^k|^{1/2}} \delta(|t| - k\ell(\gamma)) \end{aligned}$$

Here the sum runs over primitive closed geodesics and γ_0 is Euler's constant.

This is the 'geometric' side of the trace formula. The 'spectral' side of the trace formula is the Poisson formula

$$0\text{-Tr} \left[\cos \left(t \sqrt{\Delta_X - \frac{1}{4}} \right) \right] = \frac{1}{2} \sum_{s \in \mathcal{R}_X} m_s e^{-(1/2-s)|t|}$$

obtained by the authors in [76]. Guillopé and Zworski use their trace formula to obtain a lower bound on the distribution of resonances in a strip. To state it, say that $f(r) = \Omega(g(r))$ if and only if there does not exist a constant C so that $f(r) \leq Cg(r)$. They prove:

THEOREM 4.7. [77] *For any hyperbolic surface X of finite geometry and any $\varepsilon \in (0, 1/2)$, the estimate*

$$\#\{\zeta \in \mathcal{R}_X : |\zeta - 1/2| < r, \operatorname{Re}(\zeta - 1/2) > \varepsilon^{-1}\} = \Omega(r^{1-\varepsilon})$$

holds.

They conjecture that the right-hand side should be replaceable by a lower bound of the form $r^{1+\delta}$ where δ is the exponent of convergence for Γ ; see section 5.3 for results in this direction.

4.3. Counting Closed Geodesics. Selberg's zeta function encodes information on the lengths of closed geodesics of X , so it is natural to ask what information about the length spectrum (i.e., the set of lengths counted with multiplicity) may be read off from the resonance set \mathcal{R}_X . It is expected that the first zero of the zeta function (which is spectrally determined) determines the asymptotics of the counting function; this is indeed the case and can be stated in a precise way.

Let

$$N(T) = \#\{\gamma \in \mathcal{P}_X : \ell(\gamma) \leq T\}$$

and to describe the asymptotics of $N(T)$ let

$$\operatorname{li}(x) = \int_2^x \frac{dt}{\log(t)}$$

Note that

$$(4.4) \quad \operatorname{li}(x) = \frac{x}{\log x} \left(1 + \mathcal{O} \left(\frac{1}{\log x} \right) \right)$$

For a compact or finite-volume non-compact Riemann surface, it is known that

$$(4.5) \quad N(T) = \text{li}(e^T) + \sum_{k=1}^p \text{li}(e^{\alpha_k T}) + \mathcal{O}\left(e^{(3/4)T}\right)$$

as $T \rightarrow \infty$, where the α_k are determined by ‘exceptional’ eigenvalues $\lambda = \frac{1}{4} - \alpha_k^2$ of the Laplacian with $\lambda < 1/4$ (see, for example, Hejhal [80] and references therein for compact surfaces, and Sarnak [147] for the finite-volume case). The proof of (4.5) relies on Selberg’s trace formula [151].

Independently, Guillopé [69] ($n_C \geq 0$) and Lalley ($n_C = 0$) [92] showed that the counting function for closed geodesics on a non-compact hyperbolic surface (or equivalently on the compact manifold with boundary \hat{X} ; see (2.2)) has leading asymptotics

$$N(T) \sim \frac{e^{\delta T}}{\delta T}$$

where δ is the exponent of convergence of the group Γ .

Naud [119] gave a sharp result on $N(T)$ for Riemann surfaces with finite geometry and infinite volume (e.g., an infinite-volume hyperbolic surface with cusps, which is not convex co-compact). His analysis is based on Guillopé and Zworski’s work on scattering asymptotics [76] and their wave trace formula [77] (see Theorem 4.6) If $n_C > 0$ it follows that $\delta > 1/2$ (see Beardon [10]).

THEOREM 4.8. [119] *Let $X = \Gamma \backslash \mathbb{H}^2$ be geometrically finite with $n_F \geq 1$ and $\delta > 1/2$. Then, as $T \rightarrow +\infty$,*

$$N(T) = \text{li}(e^{\delta T}) + \sum_{k=1}^p \text{li}(e^{\alpha_k T}) + \mathcal{O}\left(e^{(\delta/2+1/4)T}\right)$$

where $1/2 < \alpha_p \leq \alpha_{p-1} \leq \dots \leq \alpha_0 = \delta$ and $\alpha_k(1 - \alpha_k)$ lies in the point spectrum of Δ_X .

Naud also obtains a sharper version of the asymptotic formula if $\delta \leq 1/2$ (so that $n_C = 0$). First, he uses transfer operator techniques to show that the first resonance of the resolvent, occurring at $s = \delta$, is isolated in the sense that all other resolvent resonances lie in a half plane $\text{Re}(s) \leq \delta - \varepsilon$ for some $\varepsilon > 0$ [118]. Let \mathcal{R}_X denote the set of poles of the resolvent $R_X(s)$ and let

$$\beta = \sup \{ \text{Re}(s) : s \in \mathcal{R}_X, s \neq \delta \}$$

Using this result, he is able to prove:

THEOREM 4.9. [119] *Let $X = \Gamma \backslash \mathbb{H}^2$ with $n_C = 0$, and $\delta \leq 1/2$, and let*

$$\beta^+ = \max(\beta, 0).$$

Then

$$N(T) = \text{li}(e^{\delta T}) + \mathcal{O}\left(e^{(\delta+\beta^+)T/2}\right).$$

5. Spectral Geometry of Convex Co-Compact Manifolds

We now consider a class of hyperbolic manifolds in higher dimension which are roughly analogous to the hyperbolic surfaces of infinite volume without cusps: the convex co-compact hyperbolic manifolds considered in Example 2.2. Recall that these manifolds admit a geometrically natural compactification to a manifold with boundary: $\bar{X} = X \cup M$ where M is a smooth manifold with a natural conformal

structure. A *defining function* for the boundary M is a nonnegative smooth function x on \bar{X} which is strictly positive on X and vanishes exactly to first order on M .

There are two reasons why the geometry of resonances for convex co-compact hyperbolic manifolds can be thoroughly analyzed. First, the hyperbolic metric g takes the form $g = x^{-2}\bar{g}$ where x is a defining function for $M = \partial\bar{X}$ and \bar{g} extends to a smooth nondegenerate metric on \bar{X} . In this situation the Mazzeo-Melrose construction of the resolvent [101] is available (see also [75] where a somewhat more explicit construction is given for a class of manifolds that includes the convex co-compact hyperbolic manifolds). Secondly, Selberg's zeta function (whose precise definition if $n \geq 2$ we will come to in a moment) may be studied using Ruelle's thermodynamic formalism (see Ruelle [145] and Fried [48]). These techniques together lead to a connection between resonances and classical orbits precisely analogous to the connection already discussed for hyperbolic surfaces. This connection can be exploited to study the distribution of classical orbits.

In what follows we will need a renormalized integral analogous to (4.1). If X is convex co-compact, x is a defining function for $\partial\bar{X}$, and $f \in \mathcal{C}^\infty(X)$ has a polyhomogeneous asymptotic expansion in x near $x = 0$, one can define the 0-integral of f and the 0-volume of X in analogy to (4.1) and (4.2), now taking

$$X(\varepsilon) = \{m \in X : x(m) > \varepsilon\}$$

In general, the 0-integral of a smooth function with polyhomogeneous expansion at $x = 0$ depends on the defining function and its derivatives at the boundary, but under some circumstances it yields interesting invariants. In an appendix to [130], Epstein proves that if $\dim(X) = 2m$ is even, the formula

$$(5.1) \quad 0\text{-vol}(X) = \frac{(-1)^m 2^{2m} \pi^m m!}{(2m)!} \chi(X)$$

holds, where $\chi(X)$ is the Euler characteristic of X .

5.1. Resolvent and Scattering Operator. If X is convex co-compact, it follows from Theorem 3.1 that the resolvent

$$R_X(s) = (\Delta_X - s(n-s))^{-1}$$

is a meromorphic operator-valued function for $\operatorname{Re}(s) > n/2$ (corresponding to the cut plane $\mathbb{C} \setminus [n^2/4, \infty)$) viewed as a map from $L^2(X)$ to itself. As before, it is important to study the analytic continuation of $R_X(s)$ in order to define and discuss resonances.

We view the convex co-compact manifold X as the interior of a manifold with boundary, \bar{X} , and we let x be a defining function for M . Denote by $\mathcal{C}^\infty(\bar{X})$ the space of smooth functions in \bar{X} and by $\dot{\mathcal{C}}^\infty(X)$ the space smooth functions which vanish to all orders at $x = 0$. Functions in $\mathcal{C}^\infty(\bar{X})$ have Taylor expansions in x to all orders at $x = 0$. Finally, we denote by $\mathcal{C}^\infty(X)$ the space of smooth functions on X with no growth restriction as $x \downarrow 0$. It follows from a more general result of Mazzeo and Melrose [101] (see Theorem 6.1 in what follows) that the resolvent operator $R_X(s) : \dot{\mathcal{C}}^\infty(X) \rightarrow \mathcal{C}^\infty(X)$ admits a meromorphic continuation to the complex s -plane whose poles are finite-rank smoothing operators. More precisely, the singularities of $R_X(s)$ are isolated and the Laurent expansion at each such singularity has a finite polar part whose coefficients are finite-rank smoothing operators.

The resonance set \mathcal{R}_X is the set of all singularities of $R_X(s)$. If $\zeta \in \mathcal{R}_X$ we define the multiplicity of ζ , m_ζ to be the rank of the residue of $R_X(s)$ at $s = \zeta$, i.e.,

$$(5.2) \quad m_\zeta = \dim \operatorname{Ran} P_\zeta$$

where

$$(5.3) \quad P_\zeta = \frac{1}{2\pi i} \int_\gamma (2s - n) R_X(s) ds$$

and γ is a simple closed contour enclosing ζ and no other singularity of $R(s)$. If $\operatorname{Re}(\zeta) > n/2$ then ζ is real and m_ζ is the dimension of the L^2 -eigenspace of the Laplacian with eigenvalue $\zeta(n - \zeta)$.

The scattering operator for X will play an important role in what follows, and is perhaps best understood in the framework of geometric scattering theory as formulated by Melrose [112] (especially chapter 8): see also Borthwick [12] and Joshi-Sá Barreto. [85] where the resolvent and scattering operator for asymptotically hyperbolic manifolds are analyzed in detail.

We consider the following ‘Dirichlet problem’ for generalized eigenfunctions $u \in \mathcal{C}^\infty(X)$ with boundary data $f \in \mathcal{C}^\infty(M)$:

$$(5.4) \quad \begin{aligned} (\Delta_X - s(n - s)) u &= 0 \\ u &= x^{n-s} F + x^s G \\ F|_{\partial X} &= f \end{aligned}$$

where F and G belong to $\mathcal{C}^\infty(\overline{X})$ and $\operatorname{Re}(s) = n/2$ with $s \neq n/2$. The form of the solution is dictated by the form of the Laplace operator on X : in any regular neighborhood of infinity, we can choose coordinates $(x, y) \in (0, 1) \times \mathbb{R}^n$ so that

$$\Delta_X = -(x\partial_x)^2 + n(x\partial_x) - x^2\partial_y^2$$

Indeed, the indicial operator for Δ_X is the operator

$$I(\Delta_X) = -(x\partial_x)^2 + n(x\partial_x)$$

which satisfies

$$I(\Delta_X)x^s = s(n - s)x^s$$

Using this fact, it is easy to see that for $2s - n$ not an integer, solutions having the form above (considered as power series in x) are formally determined given the boundary values of F and G at $x = 0$. In fact, for $\operatorname{Re}(s) = n/2$ with $s \neq n/2$, there is a *unique* solution to the Dirichlet problem (5.4): this follows from the absence of solutions $u \in L^2(X)$ and the ‘boundary pairing formula’ (see, for example [61] where this formula is discussed in the context of asymptotically hyperbolic manifolds). It follows from the uniqueness that the mapping

$$\begin{aligned} S_X(s) : \mathcal{C}^\infty(M) &\rightarrow \mathcal{C}^\infty(M) \\ f &\mapsto G|_M \end{aligned}$$

is well-defined. This map is called the *absolute scattering operator*. From its definition, it is not difficult to see that

$$S_X(s)S_X(n - s) = I$$

where I is the identity operator on $\mathcal{C}^\infty(M)$. The scattering operator has a meromorphic continuation to $s \in \mathbb{C}$, as was shown e.g. by Agmon [1],[2], Perry [131],

[132]. Its formal logarithmic derivative $S_X(s)^{-1}S'_X(s)$ was studied in [130]. In the next section, we discuss analogous but more general results for the scattering operator on asymptotically hyperbolic manifolds.

THEOREM 5.1. *The scattering operator $S_X(s)$ has a meromorphic continuation to \mathbb{C} with the following properties:*

(i) *For each s where it is defined, $S_X(s)$ is a pseudodifferential operator acting in suitably chosen local coordinates on M as convolution with the singular kernel*

$$c_n \frac{\Gamma(n/2 - s)}{\Gamma(s)} |x|^{-2s}$$

modulo a smoothing operator.

(ii) *At $s = n/2 + k$, $k \in \mathbb{N}$, $S_X(s)$ has a first-order pole of infinite rank.*

(iii) *At each singularity ζ of $s \mapsto S_X(s)$ in $\mathbb{C} \setminus \frac{1}{2}(n + \mathbb{N})$, the Laurent expansion of $S_X(s)$ near $s = \zeta$ has a finite polar part whose coefficients are smoothing operators.*

(iv) *The operator $S_X(s)^{-1}S'_X(s)$ has first-order poles with finite-rank residues at each singularity ζ and*

$$\nu_\zeta = -\operatorname{Tr} \left(\operatorname{Res}_{s=\zeta} [S_X(s)^{-1}S'_X(s)] \right)$$

is an integer.

The infinite rank poles of $S_X(s)$ turn out to be of great interest in conformal geometry; see Theorem 6.14.

It is important to note that, even in the model case $X = \mathbb{H}^{n+1}$, the scattering operator as we have defined it has poles of infinite rank in the ‘physical’ half-plane $\operatorname{Re}(s) > n/2$ at each of the points $s = n/2 + k$, $k = 1, 2, \dots$; see section 2 of [61] for an illuminating explanation of this phenomenon via an example from scattering theory on the half-line. The residues of $S_X(s)$ at the poles $s = n/2 + k$, $k = 1, 2, \dots$, are actually differential operators P_k of order $2k$ associated with the conformal structure on M . We will set

$$(5.5) \quad d_k = \dim \ker(P_k).$$

These operators will play a very important role in the analysis of scattering and in connections with conformal geometry.

5.2. Selberg’s Zeta Function and its Divisor. If X is convex co-compact, the discrete group Γ consists of loxodromic elements (see (2.1)), and closed geodesics of X are in one-to-one correspondence with conjugacy classes of Γ . For each conjugacy class, the length $\ell(\gamma)$ and the eigenvalues $\alpha_1(\gamma), \dots, \alpha_n(\gamma)$ of the rotation matrix A_γ are independent of the choice of representative γ . If we view the closed geodesic as a closed orbit in the cotangent bundle of X under geodesic flow, then $\ell(\gamma)$ is its period and the Poincaré once-return map \mathcal{P}_γ may be computed in terms of the invariants $\ell(\gamma), \alpha_1(\gamma), \dots, \alpha_n(\gamma)$.

Selberg’s zeta function is defined as follows. Let $\{\gamma\}$ be a listing of representatives of conjugacy classes of primitive elements of Γ . Then

$$(5.6) \quad Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{k_1, \dots, k_n \geq 0} (1 - \alpha_1(\gamma)^{k_1} \dots \alpha_n(\gamma)^{k_n}) e^{-(s+k_1+\dots+k_n)\ell(\gamma)}$$

This infinite product can be shown to converge absolutely for $\operatorname{Re}(s) > n$. It is important to note that, as $s \rightarrow \infty$ through real values, $Z_\Gamma(s)$ approaches 1 exponentially fast.

Selberg's zeta function can also be identified with the dynamical zeta function for geodesic flow on X . If γ is a primitive element of Γ , the Poincaré once-return map \mathcal{P}_γ for the associated geodesic satisfies the identity

$$|\det(I - \mathcal{P}_\gamma)|^{1/2} = e^{(n/2)\ell(\gamma)} G_\gamma$$

where

$$G_\gamma = \prod_{k=1}^n \left(1 - \alpha_j(\gamma) e^{-\ell(\gamma)}\right)$$

Elementary manipulations with (5.6) give the formula

$$Z_\Gamma(s) = \exp \left(- \sum_{m=1}^{\infty} \sum_{\gamma \in \mathcal{P}_X} \frac{1}{m} \frac{\exp(-(s - n/2)\ell(\gamma^m))}{|\det(I - \mathcal{P}_{\gamma^m})|^{1/2}} \right)$$

which exhibits the zeta function as a dynamical zeta function for geodesic flow on X . Using Ruelle's thermodynamic formalism [145] and results of Fried [48], one can prove:

THEOREM 5.2. [130] *Let Γ be an orientation-preserving, torsion-free, convex co-compact discrete group. Then Selberg's zeta function $Z_\Gamma(s)$ is a quotient of entire functions of order at most $n + 1$.*

The singularities of the zeta function are determined by the poles of the scattering operator together with the Euler characteristic of the manifold X . For $\dim(X)$ even, this result is due to Patterson and Perry [130]; for $\dim(X)$ odd, it is due to Bunke and Olbrich [23]. In what follows, $\chi(X)$ denotes the Euler characteristic of X and may be positive, negative, or zero.

THEOREM 5.3. [23],[130] *Let Γ be an orientation-preserving, torsion-free, convex co-compact discrete group, and let $X = \Gamma \backslash \mathbb{H}^{n+1}$. Selberg's zeta function $Z_\Gamma(s)$ has the following singularities:*

- (1) A zero of order m_ζ if $\zeta > n/2$ and $\zeta(n - \zeta)$ is an eigenvalue of Δ_X
- (2) A zero of order $\dim \ker(I + S_X(n/2))$ at $s = n/2$
- (3) A zero of order ν_ζ if $\operatorname{Re}(\zeta) < n/2$ ζ is a singularity of $S_X(s)^{-1} S'_X(s)$, and $\zeta(n - \zeta)$ is not an eigenvalue of Δ_X , and of order $\nu_\zeta - m_{n-\zeta}$ otherwise
- (4) A zero (or pole) of order $h_n(k)\chi(X)$ at $s = -k$ where

$$h_n(k) = (2k + n) \frac{(k + 1) \cdots (k + n - 1)}{n!}$$

Note that singularities of types (3) and (4) may coincide.

Let us explain the main ideas of the proof for $\dim(X)$ even. The key element in the proof is the identity

$$(5.7) \quad \frac{Z'_\Gamma(s)}{Z_\Gamma(s)} + \frac{Z'_\Gamma(n - s)}{Z_\Gamma(n - s)} = 0 - \operatorname{Tr}(R_X(s) - R_X(n - s)) \\ + (2\pi) \frac{(-1)^{(n+1)/2}}{n!} \frac{\Gamma(s)\Gamma(n - s)}{\Gamma(s - n/2)\Gamma(n/2 - s)} \chi(X)$$

This follows from the identity, due to Patterson [128] (and familiar in calculations with the Selberg trace formula in other settings)

$$(5.8) \quad \frac{Z'_\Gamma(s)}{Z_\Gamma(s)} = \int_{\mathcal{F}} [G_\Gamma(z, w, s) - G_0(z, w, s)]|_{z=w} \operatorname{dvol}(z)$$

for a fundamental domain \mathcal{F} in \mathbb{H}^{n+1} , where $G_\Gamma(z, w, s)$ is the lift of the resolvent kernel to \mathbb{H}^{n+1} and $G_0(z, w, s)$ is the resolvent kernel for the Laplacian on \mathbb{H}^{n+1} . Patterson's identity exploits the fact that both the logarithm of the zeta function and the resolvent can be expressed as a sum over the group Γ . To obtain (5.7) from (5.8), one uses the identity

$$G_0(z, w, s) - G_0(z, w, n - s) = \pi^{-n/2} \frac{\Gamma(s)\Gamma(n - s)}{\Gamma(s - n/2)\Gamma(n/2 - s)} \frac{\Gamma(n/2)}{\Gamma(n)}$$

(which follows from the explicit formula for $G_0(z, w, s)$) and Epstein's formula (5.1). The first right-hand term in (5.7) gives rise to zeros of the zeta function associated to eigenvalues and resolvent resonances: see (6.13), and note that the multiplicity of these zeros is determined *jointly* by the resolvent resonances and the numbers $d_k = \dim \ker P_k$, as has been emphasized by Guillarmou [62] (see Theorem 6.9 in what follows). The second right-hand term gives rise to poles whose multiplicity is determined by the Euler characteristic.

Brooks, Gornet and Perry [21] constructed pairs of non-isometric Schottky manifolds in three dimensions with the same eigenvalues and resonances, so the resonances do not uniquely determine the isometry class of a metric on a convex co-compact hyperbolic manifold.

5.3. Distribution of Resonances. Selberg's zeta function can be used to study the distribution of scattering resonances by exploiting its connection with classical dynamics. We let \mathcal{R}_X be the set of all resolvent resonances, counting multiplicity and $\mathcal{R}_{X,s}$ be the set of poles of the scattering operator. Let

$$N_X(R) = \# \{s \in \mathcal{R}_X : |s| \leq R\}$$

and

$$N_{X,s}(R) = \# \{s \in \mathcal{R}_{X,s} : |s| \leq R\}$$

In [75], Guillopé and Zworski showed that for asymptotically hyperbolic manifolds with constant curvature near infinity, a bound of the form

$$N_X(R) \leq CR^{n+2}$$

holds. This bound is not optimal in the sense that there are hyperbolic manifolds for which $N_X(r) \simeq CR^{n+1}$ (see for example the cylindrical manifolds considered in Appendix B of [130] or, for $n = 2$, the solid torus studied in [142]). Perry [140] used the results of Patterson-Perry [130] and Bunke-Olbrich [23] to prove the lower bound

$$N_{X,s}(R) \geq CR^{n+1}$$

for convex co-compact hyperbolic manifolds in any dimension. Note that the difference between the counting functions N_s and N_r depends on the numbers $\dim \ker P_k$; one expects these numbers to be zero "generically" (they are known to vanish if $\dim(X) = 2$) but on the other hand there are examples when they actually saturate the counting function (see the comments in section 2 of [68] and especially the explicit computation in Proposition 2.2 there).

A finer question is the relationship between the distribution of resonances and the 'trapped set' of geodesics. Zworski [165] proved

THEOREM 5.4. [165] *Suppose that X is a convex co-compact hyperbolic surface. For any $\alpha \in [0, 1]$ and $a, b > 0$, the estimate*

$$\# \left\{ s \in \mathcal{R}_X : \frac{1}{2} - \operatorname{Re}(s) < a |\operatorname{Im} s|^\alpha + b, |s| \leq r \right\} \leq C r^{1+\alpha+\delta(1-\alpha)}$$

holds, where δ is the Hausdorff dimension of the limit set.

Note that for $\alpha = 0$, it follows that

$$\# \left\{ s \in \mathcal{R}_X : \frac{1}{2} - \operatorname{Re}(s) < C_1, |s| \leq r \right\} \leq C_2 r^{1+\delta}$$

This gives a kind of “Weyl’s law” for resolvent resonances in strips. The dimension of the recurrent set for geodesic flow in T^*X is $2(1 + \delta)$ so that the exponent can be taken to represent the dimension of the trapped set.

Guillopé, Lin and Zworski [73] studied Selberg’s zeta function for convex co-compact Schottky groups using Ruelle transfer operators. Among other results they show that Selberg’s zeta function for such groups is entire and not simply a quotient of entire functions as shown by Patterson and Perry. They proved that, in strips parallel to the imaginary axis, the zeta function obeys the bound

$$|Z_\Gamma(s)| \leq \exp \left(C |s|^\delta \right)$$

where δ is the exponent of convergence of the Schottky group Γ . This implies the following estimate on zeros of the zeta function in strips:

THEOREM 5.5. [73] *Suppose that Γ is a convex co-compact Schottky group and let \mathcal{Z} be the set of zeros of Selberg’s zeta function, counted with multiplicity. Then for any $C_0 > 0$ there is a C_1 so that the estimate*

$$\# \{ \zeta \in \mathcal{Z} : \operatorname{Re}(s) \geq -C_0, r < |\operatorname{Im}(s)| < r + 1 \} \leq C_1 r^\delta$$

holds.

Any convex co-compact hyperbolic surface is Schottky (see section 6 of [73] and references given there). In this case, the estimate on zeros of the zeta function, combined with Theorem 5.3 imply an estimate on the distribution of resonances for convex co-compact surfaces.

THEOREM 5.6. [73] *Suppose that X is a convex co-compact hyperbolic surface. For any $C_0 > 0$ there is a constant C_1 with the property that*

$$\# \{ s \in \mathcal{R}_X : \operatorname{Re}(s) > -C_0, r \leq |\operatorname{Im}(s)| \leq r + 1 \} \leq C_1 r^\delta$$

Numerical evidence is presented suggesting that these estimates are optimal.

5.4. Counting Closed Geodesics. The counting function for closed geodesics may be studied using the zeta function much as in the surface case described in §4.3. As before, we denote by \mathcal{P}_X the set of primitive closed geodesics of X and set

$$N(T) = \# \{ \gamma \in \mathcal{P}_X : \ell(\gamma) \leq T \}$$

Perry [138] used Selberg’s zeta function together with Theorem 5.3 to prove:

THEOREM 5.7. [138] *Suppose that Γ is convex co-compact and $X = \Gamma \backslash \mathbb{H}^{n+1}$, and let δ be the exponent of convergence for Γ . Then*

$$N(T) \sim \frac{\exp(\delta T)}{\delta T}$$

as $T \rightarrow \infty$.

The key to the proof is to show that the first zero of Selberg's zeta function is, in all cases, the exponent of convergence δ , and occurs with multiplicity one. For $\delta > n/2$ this is an immediate consequence of the Patterson-Sullivan theorem [124], [159] and the characterization of the divisor of $Z(s)$. For $\delta \leq n/2$ one makes a careful study of the relationship between resolvent poles and scattering poles in the region $0 < s \leq n/2$. A similar result for certain discrete groups with parabolic elements was obtained by Dal'bo and Peigné [32] by very different methods.

Guillarmou and Naud [68] proved a more refined asymptotic formula in case X is a convex co-compact hyperbolic manifold with $\delta > n/2$. In what follows, we write eigenvalues λ_i of Δ_X as $\alpha_i(n - \alpha_i)$ where $\alpha_i > n/2$.

THEOREM 5.8. [68] *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold with $\delta > n/2$. Then*

$$N(T) = \text{li}(e^{\delta T}) + \sum_{\beta_n(\delta) < \alpha_i < \delta} \text{li}(e^{\alpha_i T}) + \mathcal{O}\left(\frac{e^{\beta_n(\delta)T}}{T}\right)$$

where $\beta_n(\delta) = (1/2 + \delta)n/(n+1)$.

The proof of this theorem rests on a formula for the 0-trace of the wave group on a convex co-compact hyperbolic manifold together with the results of Patterson-Perry [130] and Bunke-Olbrich [23] on the divisor of Selberg's zeta function. The trace formula is of interest in its own right; recall the numbers d_k defined in (5.5).

THEOREM 5.9. [68] *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold. Then, as distributions on $\mathbb{R} \setminus \{0\}$*

$$(5.9) \quad \begin{aligned} 0\text{-tr}\left(\cos\left(t\sqrt{\Delta_X - n^2/4}\right)\right) &= \frac{1}{2} \sum_{s \in \mathcal{R}_X} m_s e^{-(n/2-s)|t|} \\ &+ \frac{1}{2} \sum_{k \in \mathbb{N}} d_k e^{-|k|t} \\ &- 2^{-n-1} A(X) \frac{\cosh(t/2)}{\sinh(|t|/2)^{n+1}} \end{aligned}$$

where

$$A(X) = \begin{cases} 0 & \text{if } (n+1) \text{ is even} \\ \chi(X) & \text{if } (n+1) \text{ is odd} \end{cases}$$

We also have

$$(5.10) \quad \begin{aligned} 0\text{-tr}\left(\cos\left(t\sqrt{\Delta_X - n^2/4}\right)\right) &= \sum_{\gamma \in \mathcal{P}_X} \sum_{k=1}^{\infty} \frac{\ell(\gamma)}{2|1 - P_{\gamma^k}|} \delta(|t| - k\ell(\gamma)) \\ &+ B(X) \frac{\cosh(t/2)}{(\sinh(|t|/2))^{n+1}} \end{aligned}$$

where

$$B(X) = \begin{cases} c_n \text{ 0-vol}(X) & \text{if } (n+1) \text{ is even} \\ 0 & \text{if } (n+1) \text{ is odd} \end{cases}$$

This trace formula should be compared with Theorem 4.6. In that case, it is known that the integers d_k are all zero. The identities (5.9) and (5.10) are the 'spectral' and 'geometric' sides of the trace formula.

One can also ask how geodesics are distributed between homology classes, a question which was already studied for compact manifolds of negative curvature by Katsuda and Sunada [89] and Phillips-Sarnak [143]. Epstein [43] studied the distribution of geodesics in homology classes for hyperbolic manifolds of finite volume and any dimension. More recently, Sharp [152] (see [153] for related results) has obtained precise asymptotics for the counting function of geodesics in a homology class for finite area hyperbolic surfaces

For hyperbolic manifolds, the homology group $H_1(X, \mathbb{Z})$ is the abelianization of the fundamental group $\pi_1(X)$. We assume that $H_1(X, \mathbb{Z})$ is nontrivial and let φ be a homomorphism from $H_1(X, \mathbb{Z})$ onto \mathbb{Z}^r . Since $H_1(X, \mathbb{Z}) = \Gamma / [\Gamma, \Gamma]$ and the geodesics of X are in one-to-one correspondence with conjugacy classes of Γ , we may regard φ as a mapping from the closed geodesics to \mathbb{Z}^r and partition the closed geodesics of X into equivalence classes labelled by elements $\alpha \in \mathbb{Z}^r$. Let

$$\pi_\alpha(T) = \#\{\gamma \in \mathcal{P}_X : \ell(\gamma) \leq T, \varphi(\gamma) = \alpha\}$$

We wish to understand asymptotic behavior of $\pi_\alpha(T)$ and in particular to show ‘equipartition of geodesics’ among the homology classes.

Associated to each $\theta \in \mathbb{T}^r$ is a unitary character χ_θ of Γ given by

$$\chi_\theta(\gamma) = \exp(2\pi i \langle \theta, \varphi(\gamma) \rangle)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on \mathbb{R}^r . As in the work of previous authors, the spectral problem

$$\begin{aligned} \Delta u &= \lambda u \\ u(\gamma(z)) &= \chi_\theta(\gamma)u(z) \end{aligned}$$

plays a role. We denote by $\lambda(\theta)$ the lowest eigenvalue of the spectral problem, which is unique for small θ .

McGowan and Perry [106] prove:

THEOREM 5.10. [106] *Let Γ be a convex co-compact discrete group of isometries of \mathbb{H}^{n+1} with $\delta > n/2$. Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ and suppose there is an onto homomorphism $\varphi : H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z}^r$. Then*

$$\pi_\alpha(T) \sim \frac{\exp(\delta T)}{T^{r/2+1}} (c_0 + c_1 T^{-1} + c_2 T^{-2} + \dots)$$

where c_0 is strictly positive and depends only on r and the Hessian of $\lambda(\theta)$ at $\theta = 0$.

The proof of Theorem 5.10 uses a Selberg trace formula for the Laplacian twisted by a character, following the approach of Phillips and Sarnak. Using very different methods, Babillot and Peigné [7], [8] studied the distribution of geodesics in homology classes for hyperbolic manifolds of Schottky type with cusps; for these groups $\Gamma \backslash \mathbb{H}^{n+1}$ has infinite hyperbolic volume but may have both regular and cusp neighborhoods at infinity. Schottky groups admit a coding of the geodesic flow by methods of symbolic dynamics; the authors use this symbolic coding and Ruelle’s theory of transfer operators to obtain asymptotic formulas for $\pi_\alpha(T)$ of the form

$$\pi_\alpha(T) \sim C_\Gamma \frac{\exp(\delta T)}{P(T)}$$

where the form of $P(T)$ depends on the Hausdorff dimension δ of the limit set for Γ (see also the comments in [43] concerning equidistribution for hyperbolic manifolds of finite volume with cusps).

6. Spectral Geometry of Asymptotically Hyperbolic Manifolds

Many of the results of the preceding sections are illuminated by considering the case of asymptotically hyperbolic manifolds. To define this class, we first recall the class of conformally compact manifolds. Suppose that \overline{X} is a smooth manifold with boundary whose interior we will denote by X . We set $M = \partial\overline{X}$. A smooth Riemannian metric g on X is called *conformally compact* if there is a defining function x for M and a smooth metric \overline{g} on \overline{X} with the property that $x^2g = \overline{g}$ on X . In this case we set $h = \overline{g}|_M$. If, in addition, the differential dx satisfies the condition $|dx(p)|_{\overline{g}} = 1$ for each $p \in M$ (here $|\cdot|_{\overline{g}}$ denotes the norm on one-forms induced by the metric \overline{g}), the sectional curvatures of g approach -1 as $x \downarrow 0$, and the metric g is called *asymptotically hyperbolic*.

If x is a defining function for M , then so is $\widehat{x} = e^{2\varphi}x$ for any $\varphi \in \mathcal{C}^\infty(\overline{X})$. For this reason, the metric $h = x^2g|_M$ is determined only up to a conformal factor. Thus an asymptotically hyperbolic metric g on X determines an equivalence class $[h]$ of conformally equivalent metrics on M . The pair $(M, [h])$ is called the *conformal infinity* of (X, g) .

Suppose that (X, g) is an asymptotically hyperbolic manifold with conformal infinity $(M, [h])$. There is a collar neighborhood $(0, \varepsilon) \times M$ of the boundary and we denote by $H(x)$ a smooth function on $(0, \varepsilon)$ taking values in the symmetric two-forms on M . Graham and Lee [60] (see also [58] and section 2 of [64]) showed that for each boundary metric $h \in [h]$ there is a unique boundary defining function x so that the metric g assumes the model form

$$(6.1) \quad g = \frac{dx^2}{x^2} + \frac{H(x)}{x^2}$$

where $H(\varepsilon)$ is a metric on the hypersurface $x = \varepsilon$, and $H(0) = h$. We will call such a function a *special defining function*. We will call a metric g *even modulo x^{2k+1}* if

$$(6.2) \quad H(x) \sim \sum_{j=0}^k h_j x^{2j} + \mathcal{O}(x^{2k+1})$$

for smooth symmetric two-tensors h_j on M . Guillarmou shows that this condition is independent of the choice of special defining function (see [64], Lemma 2.1). Hyperbolic metrics are even modulo $\mathcal{O}(x^\infty)$.

A special class of asymptotically hyperbolic metrics will play an important role in what follows. Fefferman and Graham [46] showed that, given a conformal manifold $(M, [h])$ where M bounds a compact manifold \overline{X} , there is an asymptotically hyperbolic metric g with conformal infinity $[h]$ which satisfies

$$\text{Ric}(g) + ng = \begin{cases} \mathcal{O}(x^\infty), & n \text{ odd} \\ \mathcal{O}(x^{n-2}), & n \text{ even} \end{cases}$$

with an additional condition if n is even. Such metrics are called *asymptotically Einstein*. Hyperbolic metrics are exact solutions of the Einstein equation $\text{Ric}(g) + ng = 0$.

As in the case of convex co-compact hyperbolic manifolds, we can define the notion of 0-integral for a smooth function on X as

$${}^0 \int f dg = \text{FP}_{\varepsilon \downarrow 0} \int_{x > \varepsilon} f dg$$

if the finite part exists. This will be the case if f admits an asymptotic expansion near $x = 0$ of the form

$$f(x, y) \sim \sum_{j \geq -N} \sum_{k=0}^{M_j} a_{j,k}(y) x^j (-\log x)^k$$

for smooth functions a_{jk} on M . The 0-trace of an operator T with smooth kernel K is the 0-integral of its diagonal, i.e.,

$$0\text{-Tr } T =^0 \int K(m, m) dg(m)$$

In general the 0-integral and 0-trace will depend on the choice of defining function but for certain operators and appropriate classes of metrics, it leads to an invariantly defined quantity: see the papers of Albin [5], [6] and the work of Guilmarmou [67] discussed below. The 0-trace is a natural trace on the algebra of \mathcal{V}_0 -pseudodifferential operators developed in [101].

6.1. Resolvent and Scattering Operator. To discuss the spectral theory of the Laplacian we need to describe its form in a neighborhood of infinity. Suppose that (X, g) is an asymptotically hyperbolic manifold and $x^2 g|_M = h$. Choose a special defining function, as described above, so that g takes the form

$$(6.3) \quad g = x^{-2}(dx^2 + H(x)).$$

where $H(0) = h$. The Laplace operator then takes the form

$$(6.4) \quad \Delta_X = I(x\partial_x) + x^2 \Delta_{h(x)} + x \text{Tr}(h^{-1}(x)\partial_x h)(x\partial_x)$$

where

$$(6.5) \quad I(x\partial_x) = -(x\partial_x)^2 + n(x\partial_x)$$

is the indicial operator that describes the degeneracy of the Laplacian near M .

The Laplacian on an asymptotically hyperbolic manifold belongs to an algebra of pseudodifferential operators, the 0-pseudodifferential operators, studied in [101]. The development of the 0-calculus is part of a larger program of analysis on manifolds with corners which has many applications in geometric scattering theory: see [112] for an exposition and references. The importance of the 0-calculus in the present context is that it permits the inversion of elliptic operators in the calculus, including the Laplacian.

If (X, g) is an asymptotically hyperbolic manifold, the Laplacian has at most finitely many eigenvalues in the interval $[0, n^2/4)$ and no eigenvalues λ with $\lambda \geq n^2/4$ [102]. Let

$$R_X(s) = (\Delta_X - s(n-s))^{-1},$$

initially defined for $\text{Re}(s) > n/2$. To describe the meromorphic extension of the resolvent, we define the function spaces $\dot{\mathcal{C}}^\infty(X)$, $\mathcal{C}^\infty(X)$, and $\mathcal{C}^\infty(\bar{X})$ in analogy to those defined on convex co-compact hyperbolic manifolds. Mazzeo and Melrose [101] proved:

THEOREM 6.1. [101] *Suppose that (X, g) is an asymptotically hyperbolic manifold. The resolvent $R_X(s)$, viewed as an operator from $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^\infty(X)$, admits a meromorphic continuation to the complex plane with the points $s \in \frac{1}{2}(n - \mathbb{N})$ removed.*

The resonance set and the multiplicity of a resonance are defined as in section 5.3. The Mazzeo-Melrose construction gives the following important mapping property of the resolvent:

THEOREM 6.2. [101] *The resolvent $R_X(s)$ is a continuous map from $\dot{C}^\infty(X)$ to $x^s \mathcal{C}^\infty(\bar{X})$.*

If the metric g has constant sectional curvatures -1 in a neighborhood of infinity, Guillopé and Zworski [75] proved a sharper result on the meromorphic continuation.

THEOREM 6.3. [75] *Suppose that (X, g) is asymptotically hyperbolic and that g has constant curvature -1 in a neighborhood at infinity. Then the resolvent $R_X(s)$, viewed as a mapping from $\dot{C}^\infty(X)$ to $\mathcal{C}^\infty(X)$, admits a meromorphic continuation to the complex plane with isolated singularities. At each singularity ζ , the resolvent has a Laurent expansion with finite polar parts whose coefficients are finite-rank smoothing operators.*

These authors also prove the estimate

$$N_X(r) \leq C_X(1+r)^{n+2}$$

on the counting function for resonances; this estimate is not optimal in the dimension.

In his doctoral thesis, Guillarmou ([62] and [64]) illuminated the relationship between essential singularities of the resolvent and asymptotic behavior of the metric. We will say that $R_X(s)$ is *finitely meromorphic* in a region $\Omega \subset \mathbb{C}$ if $s \mapsto R_X(s)$ has no essential singularities in Ω , the poles of $R_X(s)$ form a discrete set, and the polar part of the Laurent expansion at each singularity consists of a finite series whose coefficients are finite-rank smoothing operators. Guillarmou proves:

THEOREM 6.4. [62], [64] *Suppose that (X, g) is an asymptotically hyperbolic manifold and that g is even modulo $\mathcal{O}(x^{2k+1})$. Then the resolvent extends to a finitely meromorphic function in the half-plane $\operatorname{Re}(s) > n/2 - k - 1/2$.*

In particular, if g is even modulo $\mathcal{O}(x^\infty)$ if and only if the meromorphically continued resolvent has no essential singularities. On the other hand, he shows that there are metrics which are even modulo $\mathcal{O}(x^{2k+1})$ for which the resolvent has an essential singularity at $s = (n-1)/2 - k$. (see [64], Corollary 4.3). Indeed, if one topologizes the asymptotically hyperbolic metrics by the topology inherited from $x^{-2} \mathcal{C}^\infty(\bar{X}, T^*\bar{X} \otimes T^*\bar{X})$, essential singularities of the meromorphically continued resolvent occur generically (see [64], Theorem 1.5).

To define the scattering operator for Δ_X , we proceed exactly as in section 5.1. Consider solutions $u \in \mathcal{C}^\infty(X)$ to the generalized eigenvalue problem

$$(\Delta_X - s(n-s))u = 0.$$

Note that solutions with $\operatorname{Re}(s) = n/2$ correspond to the continuous spectrum of Δ_X . A formal power series analysis based on the indicial operator (6.5) suggests that the general solution should take the form

$$u = x^{n-s}F + x^sG$$

where F and G belong to $\mathcal{C}^\infty(\bar{X})$. The Taylor series of F and G at $x = 0$ are formally determined by the functions $f = F|_M$ and $g = G|_M$. Moreover, it can be

shown that the Dirichlet problem

$$(6.6) \quad \begin{aligned} (\Delta_X - s(n-s))u &= 0 \\ u &= x^{n-s}F + x^sG \\ F|_M &= f \end{aligned}$$

has a unique solution for given $f \in \mathcal{C}^\infty(M)$ and fixed s with $\operatorname{Re}(s) = n/2$, $s \neq n/2$. Thus the mapping

$$(6.7) \quad \begin{aligned} S_X(s) : \mathcal{C}^\infty(M) &\rightarrow \mathcal{C}^\infty(M) \\ f &\mapsto G|_M \end{aligned}$$

is a well-defined linear operator from $\mathcal{C}^\infty(M)$ to itself. From its definition,

$$S_X(s)S_X(n-s) = I.$$

The operator $S_X(s)$ depends on the choice of defining function x ; if $\bar{x} = e^\varphi x$ is another defining function for $\varphi \in \mathcal{C}^\infty(\bar{X})$ with $\varphi|_M = \Upsilon$, it is not difficult to see that

$$(6.8) \quad \bar{S}_X(s) = e^{-s\Upsilon} S_X(s) e^{(n-s)\Upsilon}$$

Note that $\bar{h} = \bar{x}^2 g = e^{2\Upsilon} h$. In this sense the scattering operator is conformally covariant.

Although initially defined for $\operatorname{Re}(s) = n/2$ and $s \neq n/2$, the scattering operator extends to a meromorphic family of pseudodifferential operators. We denote by $\sigma(A)$ the principal symbol of the pseudodifferential operator A . Joshi and Sá Barreto [85] prove

THEOREM 6.5. [85] *Suppose that (X, g) is an asymptotically hyperbolic manifold, fix a defining function x for $M = \partial\bar{X}$, and let $h = x^2 g|_{x=0}$. Then the mapping $s \mapsto S_X(s)$, initially defined on the line $\operatorname{Re}(s) = n/2$ and $s \neq n/2$, extends to a meromorphic function on $\mathbb{C} \setminus \frac{1}{2}(n - \mathbb{N})$ taking values in the pseudodifferential operators on $\mathcal{C}^\infty(M)$. The operator $S_X(s)$ is a pseudodifferential operator of order $2s - n$ with principal symbol*

$$2^{n-2s} \frac{\Gamma(n/2 - s)}{\Gamma(s - n/2)} \sigma(\Delta_h^{s-n/2})$$

where Δ_h is the Laplacian on (M, h) . At each singularity in $\mathbb{C} \setminus \frac{1}{2}(n - \mathbb{N})$ the scattering operator has a Laurent expansion with finite polar point whose coefficients are smoothing operators.

Joshi and Sá Barreto use the Mazzeo-Melrose construction to obtain the meromorphic continuation for general, asymptotically hyperbolic metrics. The paper [85] also proves the following interesting result on inverse scattering.

THEOREM 6.6. [85] *Suppose that g_1 and g_2 are asymptotically hyperbolic metrics on a fixed manifold X , and that the associated scattering operators $S_1(s)$ and $S_2(s)$ have the property that $S_1(s) - S_2(s)$ is a pseudodifferential operator of order $2\operatorname{Re}(s) - n - k$. Then the metrics g_1 and g_2 have the property that $x^2(g_1 - g_2)$ vanishes to order k at $x = 0$.*

A stronger inverse result was recently proven by Sá Barreto by very different methods. Using the boundary control method of Belishev [9] and the author's study of radiation fields for scattering on asymptotically hyperbolic manifolds, Sá Barreto [146] proves:

THEOREM 6.7. [146] *Let (X_1, g_1) and (X_2, g_2) be two asymptotically hyperbolic manifolds with the same boundary $M_1 = M_2 = M$. Suppose that $S_1(s)$ and $S_2(s)$ are scattering matrices on X_1 and X_2 corresponding to special defining functions x_1 and x_2 , and suppose that $S_1(n/2 + i\lambda) = S_2(n/2 + i\lambda)$ for every $\lambda \in \mathbb{R} \setminus \{0\}$. Then, there is a diffeomorphism $\Psi : X_1 \rightarrow X_2$, smooth up to M , so that $\Psi = I$ on M and $\Psi^*g_2 = g_1$.*

Graham and Zworski [61] give an elegant proof that $S_X(s)$ admits a meromorphic continuation using Theorem 6.2; let us summarize the main ideas here. First, it is easy to construct a formal power series solution of $(\Delta_X - s(n-s))u = 0$ in a neighborhood of the boundary (where X has the structure of a product manifold $M \times (0, \varepsilon)$) having the form

$$u_1 \sim \sum_{j=0}^{\infty} a_j x^{n-s+j}.$$

Here the a_j are smooth functions on M , $a_0 = f$, and the remaining a_j are determined by recursion, using the fact that the indicial operator (6.5) satisfies

$$[I(x\partial_x) - s(n-s)]x^{n-s+j} = j(2s - n - j)x^{n-s+j}$$

This gives, on the formal level at least, a mapping $\Phi(s) : \mathcal{C}^\infty(M) \rightarrow x^{n-s}\mathcal{C}^\infty(\overline{X})$ with the property that $(\Delta_X - s(n-s))\Phi(s)f \in \mathcal{C}^\infty(X)$. Moreover, it is easy to see that this map is analytic in s away from the points $s = n/2 + k$, $k = 1, 2, \dots$. Since the resolvent $R_X(s)$ maps $\dot{\mathcal{C}}^\infty(X)$ to $x^s\mathcal{C}^\infty(\overline{X})$, we can solve the initial value problem by setting

$$(6.9) \quad u = \Phi(s)f - [R_X(s) \circ (\Delta_X - s(n-s)) \circ \Phi(s)](f)$$

Comparing (6.9) with the definition of the scattering operator (6.7) via the Dirichlet problem (6.6), we obtain the formula

$$S_X(s)f = -x^{-s} [R_X(s) \circ (\Delta_X - s(n-s)) \circ \Phi(s)](f)|_{x=0}.$$

Given the analytic continuation of the resolvent, this formula yields the analytic continuation of the scattering operator. It also connects the resolvent resonances and scattering resonances explicitly.

The scattering operator has poles at the resolvent resonances but also has poles at the points $s = n/2 + k$ where k is an integer. They arise in the factor $\Phi(s)$ above owing to the crossing of indicial roots in the formal power series solution that defines $\Phi(s)$. The residues

$$(6.10) \quad P_k = \operatorname{Res}_{s=n/2+k} S_X(s)$$

are actually elliptic differential operators of order $2k$ acting on $\mathcal{C}^\infty(M)$ whose coefficients depend on the the first $2k$ derivatives of the metric at the boundary. In all cases, these operators are conformally covariant in the sense that

$$\overline{P}_k = e^{-(n/2+k)\Upsilon} P_k e^{(n/2-k)\Upsilon}$$

if $\overline{h} = e^{2\Upsilon}h$. In the special case that g is an asymptotically Poincaré-Einstein metric, these operators are natural operators associated to the conformal structure on M : that is, they depend only on the metric $h = x^2g$ restricted to M . We will discuss them in greater depth below.

Recently, Perry and Schueth [141] showed that, given any metric on a complete Riemannian manifold (X, g) of dimension $n \geq 9$ admitting an isometric action by

$O(n)$, there are continuous families g_t of non-isometric perturbed metrics which are identical to g outside a compact set and have the same resolvent resonances and scattering resonances. In particular, there are continuous families of hyperbolic or asymptotically hyperbolic metrics on simply connected spaces of dimension $n \geq 9$ which have this property.

6.2. Trace Formulas. In this section we describe progress toward trace formulas recently obtained by Joshi-Sá Barreto [86] and Guillarmou [67].

Joshi and Sá Barreto [86] have studied the trace of the wave group for asymptotically hyperbolic manifolds. They study the solution operator $U(t)$ for the wave equation

$$\begin{aligned} u_{tt} + (\Delta_X - n^2/4) u &= 0 \\ u(x, 0) &= f(x) \\ u_t(x, 0) &= 0 \end{aligned}$$

and show that $U(t)$ is a Fourier integral operator. They show that the 0-trace of $U(t)$ exists and defines a distribution on \mathbb{R} . Note that this distribution need not be independent of the choice of defining function. Using their representation of $U(t)$ as a Fourier integral operator, they are able to prove the following analogue of Duistermaat-Guillemin's results for the trace of the wave group on a compact manifold.

THEOREM 6.8. [86] *Let (X, g) be an asymptotically hyperbolic manifold. The singular support of $0\text{-tr}(U(t))$ is contained in the set of periods of closed geodesics of (X, g) .*

More recently, Guillarmou [67] has obtained Krein-type formulas relating the 0-trace of the spectral family for the Laplacian to a renormalized determinant of the scattering operator. The interesting feature of these formulas is that there is no 'comparison operator' as in the Krein theory of the spectral shift, but instead the renormalizations are 'intrinsic' to the geometric setting.

We will sometimes be convenient to work with a renormalized scattering operator

$$\mathcal{S}_r(s) = 2^{2s-n} \frac{\Gamma(n/2 - s)}{\Gamma(s - n/2)} S_X(s)$$

for which infinite-rank poles are absent. It can be shown that the operator

$$(6.11) \quad T(s) = -\mathcal{S}_r(s)^{-1} \mathcal{S}'_r(s)$$

has at most first-order poles whose singularities are finite-rank operators; moreover, the traces

$$(6.12) \quad \nu_\zeta = \text{Tr} \left[\text{Res}_{s=\zeta} T(s) \right]$$

are integral and count the multiplicity of scattering poles (see [130] for the case of convex co-compact hyperbolic manifolds and Guillarmou [67] for the case of asymptotically hyperbolic manifolds).

The poles of the scattering operator 'essentially' correspond to the poles of the resolvent. The precise relationship was analyzed by Guillopé and Zworski [76] for hyperbolic surfaces and by Borthwick-Perry [16] (using Agmon's [4] perturbation theory of resonances) and Guillarmou [62], [65] for asymptotically hyperbolic manifolds. Borthwick and Perry also show that the resonances are generically simple,

i.e., singularity of the resolvent at the resonance is a first-order pole with rank-one residue (compare [91] where an analogous result for Schrödinger operators is proved). Recall the definitions (6.12) and (6.10), and recall that m_ζ is the multiplicity of the resolvent resonance at ζ . Borthwick-Perry [16] and Guillarmou [65] proved:

THEOREM 6.9. [16],[64] *Let (X, g) be an asymptotically hyperbolic manifold, and suppose that ζ is a resolvent resonance. Then*

$$m_\zeta - m_{n-\zeta} = \begin{cases} \nu_\zeta & \zeta \notin n/2 - \mathbb{N} \\ \nu_\zeta - \dim \ker P_k & \zeta = n/2 - k \end{cases}$$

For $\zeta \notin n/2 - \mathbb{N}$ this result was proved by Borthwick-Perry [16]; the precise description of the case $\zeta \in n/2 - \mathbb{N}$ is due to Guillarmou [65]. If $\zeta < n/2$, the term $m_{n-\zeta}$ is zero unless $\zeta(n-\zeta)$ is one of the finitely many eigenvalues of the Laplacian.

For a geometrically natural class of asymptotically hyperbolic manifolds, Guillarmou [67] has defined a renormalized determinant of the scattering operator and used it to construct an analogue of Krein's spectral shift. He considers asymptotically hyperbolic manifolds with $\dim(X)$ is even and metrics g on X which are even modulo $\mathcal{O}(x^\infty)$. In this setting, Guillarmou exploits the work of Kontsevich-Vishik [90] and Lesch [99] to define a trace for pseudodifferential operators such as $T(s)$ above which are very far from being trace class. Although we cannot fully describe the ideas here, the starting point is to consider, for an elliptic pseudodifferential operator A , the analytic trace-valued function $Z(s) = \text{Tr}(BA^{-s})$. If B is a fixed pseudodifferential operator, A is elliptic and $\text{Re}(s)$ is sufficiently large, then $Z(s)$ is well-defined and one wants to define $\text{TR}(B) = Z(0)$ by analytic continuation. This can be carried out for suitable classes of pseudodifferential operators B . There is also a determinant of $\mathcal{S}_r(s)$ which extends meromorphically to the complex plane, formally defined as follows. Noting that $\mathcal{S}_r(s)$ is actually self-adjoint for s real and sufficiently large, one can define a Ray-Singer type determinant for such s by

$$\det(\mathcal{S}_r(s)) = \exp(-\zeta_t(s, 0))$$

where

$$\zeta(s, t) = \text{Tr}(\mathcal{S}_r(s)^{-t})$$

is well-defined (with the usual trace) for t large by the ellipticity of $\mathcal{S}_r(s)$. The determinant can be analytically continued, and it can be shown that $\det(\mathcal{S}_r(s))$ has $\text{TR}[S'_r(s)\mathcal{S}_r(s)^{-1}]$ as its logarithmic derivative.

Let $G_X(x, y, s)$ be the Schwarz kernel of the resolvent operator $R_X(s)$. Then $G_X(x, y, s) - G_X(x, y, n-s)$ is actually a smooth kernel since the diagonal singularities cancel. Guillarmou [67] proves:

THEOREM 6.10. [67] *Suppose that (X, g) is an asymptotically hyperbolic manifold, that $\dim(X)$ is even, and that g is even modulo $\mathcal{O}(x^\infty)$. Then the formula*

$$(6.13) \quad 0 - \text{Tr}((n-2s)[R_X(s) - R_X(n-s)]) = \text{TR}[S'_r(s)\mathcal{S}_r(s)^{-1}],$$

where TR is the Kontsevich-Vishik trace, holds meromorphically in $s \in \mathbb{C}$.

The left-hand side of (6.13) can naturally be regarded as the derivative of a renormalized spectral density (via Stone's formula). This formula is analogous to formulas appearing in the Krein theory of the spectral shift but the renormalization here is purely geometric. The determinant has a meromorphic continuation whose

singularities are determined by the resonances of the resolvent together with the kernels of the operators P_k . More precisely:

THEOREM 6.11. [67] *Under the hypotheses of Theorem 6.10, the function $\det(S_r(s))$ extends meromorphically to \mathbb{C} . If $\zeta \in \mathbb{C}$ then the divisor of $\det(S_r(s))$ at $s = \zeta$ is given by*

$$-m_\zeta + m_{n-\zeta} - \mathbf{1}_{n/2-\mathbb{N}}(\zeta) \dim \ker S_r(n-\zeta) + \mathbf{1}_{n/2+\mathbb{N}}(\zeta) \dim \ker S_r(\zeta)$$

Here $\mathbf{1}_S$ denotes the characteristic function of the set S and a pole is assigned negative sign.

For the case of convex co-compact hyperbolic manifolds, Guillarmou proves a beautiful formula relating the determinant of the scattering operator to the zeta function.

THEOREM 6.12. [67] *Let $X = \Gamma \backslash \mathbb{H}^{n+1}$ be a convex co-compact hyperbolic manifold having even dimension, let $\chi(X)$ be the Euler characteristic of X , and let $Z_X(s)$ be Selberg's zeta function for X . Finally, let $m_{n/2}$ be the rank of the residue of $R_X(s)$ at $s = n/2$. Then*

$$\det S_r \left(\frac{n}{2} + i\lambda \right) = (-1)^{m_{n/2}} \frac{Z_X(n/2 + i\lambda)}{Z_X(n/2 - i\lambda)} \exp(-i\chi(X)G(z))$$

where

$$G(z) = (-2)^{(n+1)/2} \frac{\pi^{1/2} \Gamma(\frac{n}{2})}{n! \Gamma(n)} \int_0^z \frac{\Gamma(n/2 + it) \Gamma(n/2 - it)}{\Gamma(it) \Gamma(-it)} dt$$

6.3. Scattering Theory and Conformal Geometry. In this subsection we describe recent work of Graham and Zworski [61] which connects the scattering operator on (X, g) with invariants of the conformal structure on the conformal manifold $(M, [h])$.

First, let us briefly describe the conformal invariants studied in [61]. Graham, Jenne, Mason and Sparling [59] proved the existence of certain natural, conformally covariant differential operators on a conformal manifold. A differential operator on (M, h) is *natural* if it can be expressed in terms of the metric h , its curvature, and covariant derivatives of h . A family of operators P_h defined for $h \in [h]$ is *conformally invariant* of weight (w, w') if

$$P_{\bar{h}} = e^{w'\Upsilon} P_h e^{-w\Upsilon}$$

for $\bar{h} = e^{2\Upsilon} h$. Note that, in this sense, the scattering operator $S_X(s)$ for an asymptotically hyperbolic manifold is a conformally covariant pseudodifferential operator of weight $(s - n, -s)$

THEOREM 6.13. [59] *Let $(M^n, [h])$ be a conformal manifold. For $k \in \mathbb{N}$ (or $1 \leq k \leq n/2$ if n is even), there exist natural, conformally covariant differential operators P_k of weight $(k - n/2, -n/2 - k)$ so that the leading symbol $\sigma(P_k)$ is the principal symbol of Δ_h^k .*

These ‘conformally invariant powers of the Laplacian’ (often referred to in the literature as the ‘GJMS operators’) play a fundamental role in conformal geometry. The operator

$$P_1 = \Delta_h + \frac{n-2}{4(n-1)} R$$

where R is the scalar curvature, is the Yamabe operator (see, for example, Lee and Parker [98]): it occurs in the study of conformal deformations of metrics to a metric of constant scalar curvature. In even dimensions, Branson's Q -curvature [19] is an analogue in conformal geometry of the scalar curvature (indeed, in two dimensions, $Q = R/2$); it transforms under conformal deformations $h \mapsto e^{2\Upsilon} h$ as

$$e^{n\Upsilon} \widehat{Q} = Q + P_{n/2} \Upsilon$$

and its integral is a conformal invariant.

Graham and Zworski [61] proved:

THEOREM 6.14. [61] *Let $(M^n, [h])$ be a conformal manifold and let (X, g) be a Poincaré metric associated to $[h]$. Suppose that $k \in \mathbb{N}$ if n is odd, or $k \leq n/2$ if n is even. Finally, suppose that $(n/2)^2 - k^2$ is not an eigenvalue of the Laplacian Δ_X . Then the scattering operator $S_X(s)$ has a simple pole at $s = n/2 + k$ and*

$$c_k P_k = - \operatorname{Res}_{s=n/2+k} S_X(s)$$

where the P_k are the GJMS operators and

$$c_k = \frac{(-1)^k}{2^{2k} k! (k-1)!}$$

Moreover, if n is even, $P_{n/2} 1 = 0$ and

$$c_{n/2} Q = S_X(n) 1$$

The authors deduce the self-adjointness of the P_k from the self-adjointness of $S_X(s)$ for real s . A different analysis, using formal power series arguments in the place of scattering theory, was later given by Graham and Fefferman [47].

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(perry) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY
40506-0027

E-mail address, perry: `perry@ms.uky.edu`