Solutions of mKdV in classes of functions unbounded at infinity

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Abstract

We investigate the relation between the Korteweg - de Vries and modified Korteweg - de Vries equations (KdV and mKdV), and find a new algebro-analytic mechanism, similar to the Lax L-A pair, which involves a first-order operator Q instead of the third-order operator A. In our framework, eigenfunctions of the Schrödinger operator L, whose time-dependent potential solves the KdV equation, evolve according to a linear first-order partial differential equation, giving explicit control over their time evolution. As an application, we establish global existence and uniqueness for solutions of the initial value problem for mKdV in classes of smooth functions which can be unbounded at infinity, and may even include functions which tend to infinity with the space variable. We also establish invariance of the spectrum and unitary type of L under the KdV flow and the invariance of the spectrum and unitary type of the impedance operator under the mKdV flow for potentials in these classes. We also show that generalized eigenfunctions of L evolve in time according to a first order partial differential equation, depending on the spectral parameter.

1 Introduction

The purpose of this work is to solve the *modified Korteweg - de Vries equation* (mKdV) on the line

$$r_t - 6r^2 r_x + r_{xxx} = 0 \tag{1}$$

$$r|_{t=0} = r_0 \tag{2}$$

in various classes of smooth functions (possibly) unbounded at $+\infty$ and/or $-\infty$. Equation (1) is closely related to the celebrated Korteweg - de Vries equation (KdV),

$$q_t - 6qq_x + q_{xxx} = 0 \tag{3}$$

and is a model equation for wave propagation.

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Let $I = (a, b) \subseteq \mathbb{R}$ with $-\infty \le a < b \le \infty$. For any given $\beta \in \mathbb{R}$ denote by $\mathcal{S}_\beta(I \times \mathbb{R})$ the linear space of $C^\infty(I \times \mathbb{R})$ functions having asymptotic expansions at $+\infty$ and $-\infty$ (cf. [3])

$$r(t, x) \sim \sum_{k=0}^{\infty} a_k^+(t) x^{\beta_k} \quad \text{as} \quad x \to \infty \tag{4}$$

and

$$r(t, x) \sim \sum_{k=0}^{\infty} a_k^-(t)(-x)^{\beta_k} \quad \text{as} \quad x \to -\infty \tag{5}$$

where $a_k^\pm \in C^\infty(I)$ and $\beta = \beta_0 > \beta_1 > ...$ with $\lim_{k\to\infty} \beta_k = -\infty$. By definition, the relations (4) and (5) mean that for any compact interval $J \subseteq I$ and any $N \ge 0$, $i, j \ge 0$, there exists a constant $C_{J,N,i,j} > 0$ such that for any $\pm x \ge 1$ and $t \in J$

$$\Big| \partial_t^i \partial_x^j \Big(r(t, x) - \sum_{k=0}^{N} a_k^\pm(t)(\pm x)^{\beta_k} \Big) \Big| \le C_{J,N,i,j} |x|^{\beta_{N+1} - j}. \tag{6}$$

For an arbitrarily chosen formal series $\sum_{k=0}^{\infty} a_k^\pm(t)(\pm x)^{\beta_k}$, referred to as a symbol in the theory of pseudodifferential operators, there exists a function $r \in C^\infty(I \times \mathbb{R})$ satisfying (4) and (5) (see for example [16, Proposition 3.5]). Analogously one defines the linear space $\mathcal{S}_\beta(\mathbb{R})$ as the space of functions $r \in C^\infty(\mathbb{R})$ having asymptotic expansions $r(x) \sim \sum_{k=0}^{\infty} a_k^\pm(\pm x)^{\beta_k}$ as $x \to \pm\infty$ where $a_k^\pm$ are given constants, $\beta = \beta_0 > \beta_1 > ...$ and $\lim_{k\to\infty} \beta_k = -\infty$.

In this paper we first prove the following results about the initial value problem (1)-(2).

Theorem 1.1. *For any $\beta < 1/2$ and for any initial data $r_0 \in \mathcal{S}_\beta(\mathbb{R})$ there exists a global in time solution $r \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (1)-(2). The solution r is unique in the class of solutions of (1)-(2) in $\mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$. Moreover, the coefficients $a_0^\pm(t)$ in the asymptotic expansion of the solution $r(t, x)$ are independent of t and are equal to the coefficients $a_0^\pm$ in the asymptotic expansion of the initial data r_0.*

By the same method of proof we obtain similar results for the larger spaces of functions $\mathcal{O}_\beta(I \times \mathbb{R})$ and $o_\beta(I \times \mathbb{R})$ which are (possibly) unbounded at infinity.

Let $I = (a, b) \subseteq \mathbb{R}$ with $-\infty \le a < b \le \infty$. For any given $\beta \in \mathbb{R}$ denote by $\mathcal{O}_\beta(I \times \mathbb{R})$ the linear space of functions $r(t, x)$ in $C^\infty(I \times \mathbb{R})$ such that for any compact interval $J \subseteq I$ and any $k, l \ge 0$ there exists a constant $C_{J,k,l} > 0$ such that for any $|x| \ge 1$ and any $t \in J$

$$|\partial_t^k \partial_x^l r(t, x)| \le C_{J,k,l} |x|^{\beta - l}.$$

Analogously one defines the linear space $\mathcal{O}_\beta(\mathbb{R})$ as the space of functions $r(x)$ in $C^\infty(\mathbb{R})$ such that for any $l \ge 0$ there exists $C_l > 0$ such that for any $|x| \ge 1$, $|\partial_x^l r(x)| \le C_l |x|^{\beta - l}$.

We will also consider the following spaces. For any given $\beta \in \mathbb{R}$ denote by $o_\beta(I \times \mathbb{R})$ the linear space of functions $r(t,x)$ in $C^\infty(I \times \mathbb{R})$ such that for any compact interval $J \subseteq I$ and any $k, l \geq 0$

$$\partial_t^k \partial_x^l r(t,x) = o(|x|^{\beta - l})$$

uniformly in $t \in J$. In the same way as above one defines the space $o_\beta(\mathbb{R})$. Clearly the following inclusions hold:

$$\mathcal{S}_\beta(I \times \mathbb{R}) \subseteq \mathcal{O}_\beta(I \times \mathbb{R}), \quad o_\beta(I \times \mathbb{R}) \subseteq \mathcal{O}_\beta(I \times \mathbb{R}).$$

Theorem 1.2. *For any $\beta < 1/2$ and for any initial data $r_0 \in \mathcal{O}_\beta(\mathbb{R})$ there exists a global in time solution $r \in \mathcal{O}_\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (1)-(2). The solution r is unique in the class of solutions of (1)-(2) in $\mathcal{O}_\beta(\mathbb{R} \times \mathbb{R})$.*

Theorem 1.3. *For any $\beta \leq 1/2$ and for any initial data $r_0 \in o_\beta(\mathbb{R})$ there exists a global in time solution $r \in o_\beta(\mathbb{R} \times \mathbb{R})$ of the initial value problem (1)-(2). The solution r is unique in the class of solutions of (1)-(2) in $o_\beta(\mathbb{R} \times \mathbb{R})$.*

Remark 1.4. *Note that for $r_0 \in \mathcal{S}_\beta(\mathbb{R})$ with $\beta = \beta_0 > 1/2$, with an asymptotic expansion of the form*

$$r_0(x) \sim \sum_{k=0}^{\infty} a_k^+ x^{\beta_k} \quad \text{as } x \to +\infty$$

with $a_0^+ \neq 0$, no formal solution and therefore no solution of mKdV in $\mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$ exists.

Remark 1.5. *In fact, the uniqueness in all Theorems 1.1, 1.2, 1.3 holds if we only require $r \in o_{1/2}(\mathbb{R} \times \mathbb{R})$ which is the largest class where the existence is claimed in these theorems. So if we only require $r \in o_{1/2}(\mathbb{R} \times \mathbb{R})$ and take the initial condition r_0 in $\mathcal{S}_\beta(\mathbb{R})$, $\mathcal{O}_\beta(\mathbb{R})$ ($\beta < 1/2$) or $o_\beta(\mathbb{R})$ ($\beta \leq 1/2$), then we will automatically have $r \in \mathcal{S}_\beta(\mathbb{R} \times \mathbb{R})$, $\mathcal{O}_\beta(\mathbb{R} \times \mathbb{R})$ or $o_\beta(\mathbb{R} \times \mathbb{R})$ respectively.*

Similar results for KdV as the ones stated for mKdV in Theorem 1.1, 1.2, and 1.3 have been obtained in a series of papers [2, 3, 4, 5] – see Appendix B where for the convenience of the reader, we give a short summary of these results. In fact, we construct our solutions of mKdV with the properties stated in the above theorems by applying to the solutions of [2, 3, 4, 5] an inverse of the Miura map. Recall that the Miura map $r \mapsto B(r) := r_x + r^2$, first introduced in [15], maps smooth solutions of mKdV to smooth solutions of KdV. However, the Miura map is usually neither 1-1 nor onto. This is, for example, the case when B is considered as a map $H^\beta_{loc}(\mathbb{R}) \to H^{\beta-1}_{loc}(\mathbb{R})$ with $\beta \geq 0$ [9]. In this case, the preimage of an element in $H^{\beta-1}_{loc}(\mathbb{R})$ is either the empty set, a point or a set homeomorphic to an interval. To describe the preimage $B^{-1}\{B(r)\}$ of $q = B(r)$, note that the positive function $\psi(x) = e^{\int_0^x r(s)ds}$ satisfies

$$-\psi_{xx} + (r_x + r^2)\psi = 0 \tag{7}$$

and is related to r by $r = \psi_x/\psi$. It has been shown in [9] that for $r \in H^\beta_{loc}(\mathbb{R})$ given with $\beta \geq 0$, any function in the preimage $B^{-1}\{B(r)\}$ arises in this way, i.e. for any $r \in H^\beta_{loc}(\mathbb{R})$,

$$B^{-1}\{B(r)\} = \{\psi_x/\psi \mid \psi \in H^\beta_{loc}(\mathbb{R}) \text{ positive, satisfying (7)}\}.$$

Given initial data r_0 in the class of functions considered in the theorems above, $q_0 = B(r_0)$ has the growth condition at infinity required by the theorems in [2, 3, 4, 5] to conclude that there exists a unique solution $q(t, x)$ of KdV in the corresponding class with $q(0, \cdot) = q_0$. We then consider the linear evolution equation

$$\psi_t(t, x) = Q(t)\psi(t, x) \tag{8}$$
$$\psi(0, x) = e^{\int_0^x r_0(s)ds} \tag{9}$$

where $Q(t)$ is the first order differential operator,

$$Q(t) := 2q(t, x)\partial_x - q_x(t, x). \tag{10}$$

and prove that there exists a unique, globally (in time) defined solution $\psi(t, x)$, satisfying $\psi(t, x) > 0$ for any $x \in \mathbb{R}$, $t \in \mathbb{R}$ and

$$-\psi_{xx}(t, x) + q(t, x)\psi(t, x) = 0. \tag{11}$$

The latter identity is shown by using the commutator relation

$$\dot{L} = [Q, L] + 4q_x L$$

where

$$L(t) := -\partial_x^2 + q(t, x). \tag{12}$$

and $\dot{L} = q_t$. The function

$$r(t, x) := \psi_x(t, x)/\psi(t, x) \tag{13}$$

is then the unique solution of mKdV with $r(0, \cdot) = r_0$ in a class of functions in $C^\infty(\mathbb{R} \times \mathbb{R})$ satisfying appropriate growth conditions. It has the claimed properties in each of the settings of Theorem 1.1, 1.2, and 1.3. We call the pair of operators (Q, L), satisfying the conditions above a Q-L pair. Such a pair allows us to construct an inverse of the Miura map and, in this way, deduce existence and uniqueness of solutions for (1)-(2) from the corresponding results for KdV.

We also establish the invariance of the spectrum of the Schrödinger operator under the KdV flow and the invariance of the spectrum of the impedance operator under the mKdV flow. Consider the Schrödinger operator $L(t) = -\frac{d^2}{dx^2} + q(t, x)$ where $q(t, x)$ is a solution of the KdV equation in $\mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \leq 1$ and $I = (a, b)$, $-\infty \leq a < b \leq +\infty$. (By Sears theorem (cf. [1, Chapter II]), for any given $t \in (a, b)$ the operator $-\frac{d^2}{dx^2} + q(t, x)$ with domain $C_0^\infty(\mathbb{R})$ is essentially self-adjoint. Denote by $L(t)$ its closure.) We will prove the following

Theorem 1.6. *Let $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \leq 1$ be a solution of the KdV equation. Then*

(i) For any $t, t' \in I$

$$\mathtt{spec}\, L(t) = \mathtt{spec}\, L(t') \,.$$

Moreover, the point spectrum (i.e. eigenvalues corresponding to L^2-eigenfunctions) of the operators coincide and have the same multiplicity.

(ii) If $\beta < 1$ or $q \in o_\beta(I \times \mathbb{R})$ with $\beta \leq 1$, the operators $L(t)$ and $L(t')$ are unitarily equivalent.

Remark 1.7. *Our method suggests that the operators $L(t)$ and $L(t')$ are unitarily equivalent also in the case when $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \leq 1$. The proof of this statement will require an improvement of Theorem 2 in [4].*

In order to prove the first statement of Theorem 1.6 we again use the Q-L formalism, now in an extended form with spectral parameter. For the proof of the second statement we use the classical Lax pair. As a corollary of Theorem 1.6 we deduce that the mKdV flow in $\mathcal{O}_\delta(I \times \mathbb{R})$ with $\delta \leq 1/2$ preserves the spectrum of the impedance operator (see Theorem 5.3). In addition, we prove Theorem 6.2 which states that the linear evolution corresponding to the first order differential operator $Q_\lambda(s) := (4\lambda + 2q(s,x))\partial_x - q_x(s,x)$, where s is a real parameter between t and t' and λ runs the set of all generalized eigenvalues of $L(t)$, transforms any complete orthogonal system of generalized eigenfunctions of the operator $L(t)$ to a complete orthogonal system of generalized eigenfunctions of the operator $L(t')$. Moreover, by Proposition 6.7, the solution of the evolution equation involving the third order differential operator $A(t) := -4\partial_x^3 + 6q(t,x)\partial_x + 3q_x(t,x)$ appearing in the classical Lax pair, can be obtained in terms of the solution of the first order evolution equation $\psi_t = Q_\lambda(t)\psi$ with spectral parameter $\lambda \in \mathbb{R}$.

Growing solutions of evolution equations such as KdV and mKdV require the development of new techniques for their study and are of interest by themselves. They recently attracted a lot of attention. In [6], Dubrovin studied Hamiltonian perturbations of the (simplest) hyperbolic equation $u_t + a(u)u_x = 0$ in one space dimension. He conjectured that the behavior of a solution to the perturbed equation near a point where the gradient of the corresponding solution of the unperturbed equation blows up, is *universal*. This means that the behavior is (essentially) independent of the choice of the (generic) Hamiltonian perturbation and of the (generic) solution of the perturbed equation. In fact, he conjectured that the behavior of solutions of the perturbed equations near such points is described by a special smooth globally (in X, T) defined solution $U(X,T)$ of an integrable 4th order ODE in the variable X which depends on a (real) parameter T. When viewed as function of X and T, $U(X,T)$ satisfies the KdV equation and grows for $X \to \pm\infty$ as $\mp(6|X|)^{1/3}$.

Related work: Beside the works [2, 3, 4, 5], we would like to mention earlier work on unbounded solutions of KdV by Menikoff [14] as well as work of Kenig, Ponce, and Vega [11]. Menikoff showed that for initial data in $o_1(\mathbb{R})$, KdV can be solved in $C^\infty(\mathbb{R}\times\mathbb{R})$ whereas Kenig, Ponce, and Vega studied solutions of KdV in special classes of unbounded functions, different from the ones considered in this paper. It was pointed out in [14] that the KdV flow with initial data in $o_1(\mathbb{R})$ preserves the discrete spectrum of the Schrödinger operator $L(t)$. We remark that the Miura map has been used previously to obtain solutions of mKdV from solutions of KdV. In particular, we mention the paper [10] where periodic solutions of low regularity are obtained, and work of Gesztesy–Simon [8] and Gesztesy–Schweiger–Simon [7] for bounded solutions of mKdV. In [7], the existence of solutions $r(t,x)$ of mKdV is proved under the assumption that $q(t,x), q_x(t,x)$ and $q_t(t,x)$ belong to $L^\infty(\mathbb{R}\times\mathbb{R})$. Rather then solving the evolution equation (8)-(9), induced by the first order operator $Q(t)$ in (10) to obtain a representation of a solution $r(t,x)$ of the form (13), the authors of [7] use the operator $A(t) = -4\partial_x^3 + 6q(t,x)\partial_x + 3q_x(t,x)$, appearing in the Lax pair for KdV. It turns out that the third order operator A is not well suited to prove the existence of a solution $r(t,x)$ of mKdV in the classes of increasing functions considered above.

The operator Q was considered by Marchenko [13] and also appears in [8, 7]. But to the best of our knowledge, in this paper, the existence of solutions of the evolution equation (8)-(9) and their properties are used for the first time as an analytic tool to study KdV and mKdV.

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2 Q-L pair

Suppose that $q \in C^\infty(\mathbb{R}\times\mathbb{R})$ and consider the differential operators $Q(t), L(t)$ given by (10) and (12), respectively.

Lemma 2.1. *The operators Q and L satisfy the following commutator relation*

$$\dot{L} = [Q, L] + 4q_x L + KdV(q) \tag{14}$$

where $\dot{L} = q_t(t,x)$ and $KdV(q) = q_t - 6qq_x + q_{xxx}$. In particular,

$$KdV(q) = 0 \quad \textit{iff} \quad \dot{L} = [Q, L] + 4q_x L. \tag{15}$$

The proof of the lemma is straightforward.

Assume that $q \in C^\infty(\mathbb{R}\times\mathbb{R})$ satisfies the KdV equation and that for any $T > 0$ there exists a constant $C_T > 0$ such that for any $|x| \geq 1$ and $t \in [-T, T]$

$$|q(t,x)| \leq C_T |x| \ . \tag{16}$$

Let $\psi_0 \in C^\infty(\mathbb{R})$ be an eigenfunction of $L(0)$ with eigenvalue 0, i.e.

$$L(0)\psi_0 = 0 \ . \tag{17}$$

Consider the equation

$$\begin{aligned} \psi_t(t,x) &= Q(t)\psi(t,x) &\quad (18)\\ \psi|_{t=0} &= \psi_0. &\quad (19) \end{aligned}$$

By Lemma A.1, the initial value problem (18)-(19) has a unique solution $\psi(t,x)$ in $C^\infty(\mathbb{R}\times\mathbb{R})$.

Proposition 2.2. *If $q \in C^\infty(\mathbb{R}\times\mathbb{R})$ is a solution of KdV satisfying the growth condition (16), and $\psi(t,x) \in C^\infty(\mathbb{R}\times\mathbb{R})$ solves (18)-(19), then*

$$L(t)\psi(t,x) = 0 \ \ \forall t, x \in \mathbb{R}. \qquad (20)$$

If, in addition, $\psi_0(x) > 0\ \forall x \in \mathbb{R}$, then $\psi(t,x) > 0\ \forall x, t \in \mathbb{R}$.

Proof. Let $\varphi(t,x) := L(t)\psi(t,x)$. It follows from (17) that $\varphi|_{t=0} = 0$. Using Lemma 2.1 and (18) one obtains

$$\begin{aligned} \varphi_t &= \dot{L}\psi + L\psi_t\\ &= ([Q,L] + 4q_x L)\psi + LQ\psi\\ &= Q(L\psi) + 4q_x(L\psi)\\ &= 2q\varphi_x + 3q_x\varphi\ . &\quad (21) \end{aligned}$$

Hence $\varphi = \varphi(t,x)$ is a solution of the initial value problem

$$\begin{aligned} \varphi_t(t,x) &= 2q(t,x)\varphi_x(t,x) + 3q_x(t,x)\varphi(t,x)\\ \varphi|_{t=0} &= 0\ . \end{aligned}$$

Applying again Lemma A.1 we obtain that $\varphi \equiv 0$.

The last statement of the proposition follows immediately from claim (b) of Lemma A.1. □

Proposition 2.3. *Assume that $r \in C^\infty(\mathbb{R}\times\mathbb{R})$ is a solution of the initial value problem (1)-(2) for the mKdV equation and define*

$$\rho(t,x) := \rho_0(t)e^{\int_0^x r(t,s)\,ds} \qquad (22)$$

with normalizing factor $\rho_0(t)$ given by

$$\rho_0(t) := e^{\int_0^t (2r^3 - r_{xx})|_{(\tau,0)}\,d\tau}\ . \qquad (23)$$

Then $\psi(t,x) := \rho(t,x)$ is a solution of (18)-(19), where $Q(t) = 2q\partial_x - q_x$, $q = r_x + r^2$, and $\psi_0(x) := e^{\int_0^x r_0(s)\,ds}$. If, in addition, $q = r_x + r^2$ satisfies the growth condition (16) then $\rho(t,x)$ is the unique solution of (18)-(19) in $C^\infty(\mathbb{R}\times\mathbb{R})$.

Proof of Proposition 2.3. Using that $q = r_x + r^2$, one easily sees that $\rho(t,x)$ satisfies the equation $L(t)\rho = 0$. Differentiating the latter identity with respect to t and using Lemma 2.1 together with the fact that $q = r_x + r^2$ satisfies KdV (cf. [15]), we obtain

$$\begin{aligned} 0 &= \dot{L}\rho + L\rho_t \\ &= ([Q,L] + 4q_x L)\rho + L\rho_t. \end{aligned}$$

Using that $L(t)\rho = 0$, one then gets

$$0 = -L(Q\rho) + L\rho_t = L(\rho_t - Q\rho). \tag{24}$$

Hence, with $f(t,x) := \rho_t - Q\rho$ one has for any $t, x \in \mathbb{R}$

$$-f_{xx}(t,x) + q(t,x)f(t,x) = 0. \tag{25}$$

A direct computation shows that

$$f(t,0) = 0 \text{ and } f_x(t,0) = 0 \ \forall t \in \mathbb{R}. \tag{26}$$

By the uniqueness of the solutions of (25)-(26) for any fixed $t \in \mathbb{R}$, we conclude that $f(t,x) \equiv 0$, and therefore $\rho_t = Q\rho$. The uniqueness of the solution ρ follows from Lemma A.1 together with the assumption that $q(t,x)$ satisfies the growth condition (16). □

Corollary 2.4. *Assume that $q \in C^\infty(\mathbb{R} \times \mathbb{R})$ solves the KdV equation and satisfies the growth condition (16). Let $\phi, \psi \in C^\infty(\mathbb{R} \times \mathbb{R})$ be two solutions of (18) with initial data $\phi|_{t=0} = \phi_0$ and $\psi|_{t=0} = \psi_0$ respectively where $L(0)\phi_0 = 0$ and $L(0)\psi_0 = 0$. Then the Wronskian $W(\phi,\psi) := \phi\psi_x - \psi\phi_x$ is independent of $t, x \in \mathbb{R}$.*

Proof. As $\phi(t,x)$ and $\psi(t,x)$ satisfy (20) (see Proposition 2.2) we get that the Wronskian W is independent of $x \in \mathbb{R}$. Using that $\phi_{xx} = q\phi$ and $\psi_{xx} = q\psi$ one obtains

$$\begin{aligned} W_t &= \phi_t\psi_x + \phi(\psi_t)_x - \psi_t\phi_x - \psi(\phi_t)_x \\ &= (2q\phi_x - q_x\phi)\psi_x + \phi(2q\psi_x - q_x\psi)_x - (2q\psi_x - q_x\psi)\phi_x \\ &\quad -\psi(2q\phi_x - q_x\phi)_x \\ &= 0\,. \end{aligned}$$

□

Theorem 2.5. *Consider the initial value problem (1)-(2) for the mKdV equation with smooth initial data $r_0 \in C^\infty(\mathbb{R})$. Suppose that the solution $q = q(t,x)$ of the KdV equation (3) with the initial data $q|_{t=0} = q_0 := r_0' + r_0^2$ is defined globally in time, $q \in C^\infty(\mathbb{R} \times \mathbb{R})$, and satisfies the growth condition (16). Then*

(a) *the evolution equation (18)-(19) has a unique, globally defined positive solution $\psi(t,x) > 0$ and the function $r(t,x) = \psi_x(t,x)/\psi(t,x)$ is a global solution of the mKdV initial value problem (1)-(2);*

(b) *if $r_1, r_2 \in C^\infty(\mathbb{R} \times \mathbb{R})$ are solutions of the initial value problem of mKdV (1)-(2) both having q as their image with respect to the Miura map $r \mapsto r_x + r^2$ (i.e., $\forall t, x \in \mathbb{R}$, $r_{1x}(t,x) + r_1^2(t,x) = r_{2x}(t,x) + r_2^2(t,x)$), then $r_1 \equiv r_2$.*

Remark 2.6. *Loosely speaking, statement* (b) *of Theorem 2.5 says that whenever KdV has a unique solution within a certain class then mKdV has a unique solution within the corresponding class defined by the Miura map.*

Proof of Theorem 2.5. (a) Introduce

$$\psi_0(x) = e^{\int_0^x r_0(s)\, ds}. \tag{27}$$

Clearly, $\psi_0(x) > 0$ $\forall x \in \mathbb{R}$. As $q_0 = r_0' + r_0^2$ one obtains from (27) that $L(0)\psi_0 = 0$. By Proposition 2.2 the solution $\psi(t,x)$ of (18)-(19) in $C^\infty(\mathbb{R}\times\mathbb{R})$ satisfies $L(t)\psi(t,x) = 0$ $\forall t, x \in \mathbb{R}$. Moreover, $\psi(t,x) > 0$ $\forall t, x \in \mathbb{R}$. Consider the smooth function $r(t,x)$ given by (13). It follows from (27) that $r|_{t=0} = r_0$. Taking into account that $L(t)\psi(t,x) = 0$ one proves by a straightforward calculation that

$$\begin{aligned} mKdV(r) &:= r_t - 6r^2 r_x + r_{xxx} \\ &= -(\psi_t\psi_x - \psi\psi_{xt} - 6q\psi_x^2 + 3\psi_{xx}^2 + 4\psi_x\psi_{xxx} - \psi\psi_{xxxx})/\psi^2 \end{aligned}$$

(See also formula (7.42) in [7].) Using that $\psi_t = Q\psi$ one gets that $mKdV(r) = 0$. This proves claim (a).

Claim (b) follows from Proposition 2.3, as the two solutions r_1, r_2 lead to the same operator Q (cf. (10)) and the same initial data ψ_0 (cf. (19)). Indeed, as r_1 and r_2 are solutions of (1)-(2) and $q = r_{1x} + r_1^2 = r_{2x} + r_2^2$ we get from Proposition 2.3 that for $k = 1, 2$,

$$\rho_k(t,x) := \rho_{k,0}(t) e^{\int_0^x r_k(t,s)\, ds} \quad \text{with} \quad \rho_{k,0}(t) := e^{\int_0^t (2r_k^3 - (r_k)_{xx})|_{(\tau,0)}\, d\tau}$$

are solutions of the linear initial value problem (18)-(19) with the same initial data $\psi_0(x) = \rho_k(0,x) = e^{\int_0^x r_0(s)\, ds}$. As q satisfies the growth condition (16) the solution of (18)-(19) is unique and therefore $\rho_1 \equiv \rho_2$. In particular, $r_1 = \frac{\rho_{1x}}{\rho_1} = \frac{\rho_{2x}}{\rho_2} = r_2$. □

3 Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. In the sequel we will need the classes

$$\mathcal{S}_{\beta+1}^*(\mathbb{R}\times\mathbb{R}) := \{f \in C^\infty(\mathbb{R}\times\mathbb{R}) \,|\, f_x \in \mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})\}.$$

where β is a given real number. Note that the operator of integration, $f(t,x) \mapsto \int_0^x f(t,s)ds$, maps $\mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ to $\mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ whereas the operator of differentiation, $f(t,x) \mapsto \partial_x f(t,x)$, maps $\mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ to $\mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ for any $\beta \in \mathbb{R}$. Analogously one defines $\mathcal{S}^*_{\beta+1}(\mathbb{R})$.

The following Lemma describes the functions from $\mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ in terms of their asymptotics at $\pm\infty$.

Lemma 3.1. *$f \in \mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ if and only if $f \in C^\infty(\mathbb{R}\times\mathbb{R})$ and it has an asymptotic expansion for $x \to \pm\infty$ of the form*

$$f(t,x) \sim \begin{cases} \sum_{k=0}^\infty a_k^\pm(t)\,(\pm x)^{\beta_k+1} + a_*^\pm(t)\log(\pm x) & \text{if} \quad \beta+1 \geq 0 \\ c^\pm(t) + \sum_{k=0}^\infty a_k^\pm(t)\,(\pm x)^{\beta_k+1} & \text{if} \quad \beta+1 < 0 \end{cases} \tag{28}$$

where $\beta = \beta_0 > \beta_1 > \ldots$ with $\lim_{k\to\infty} \beta_k = -\infty$ and $a_k^\pm$, $a_*^\pm$, and $c_\pm$ are functions of t in $C^\infty(\mathbb{R})$. The same result holds in $\mathcal{S}^*_{\beta+1}(\mathbb{R})$.

In particular, if $\beta + 1 \geq 0$ then the leading term of the asymptotic of f is $a_0^\pm(t)(\pm x)^{\beta+1}$ (for $\beta > -1$) or $a_*^\pm(t)\log(\pm x)$ (for $\beta = -1$). If $\beta+1 < 0$, the leading term is $c^\pm(t)$ followed by $a_0^\pm(t)(\pm x)^{\beta+1}$. The asymptotic relations should be understood similarly to (4), (5). For example, the first relation in (28) means that for any compact interval $J \subseteq \mathbb{R}$, $i,j \geq 0$, and any $N \geq 0$ with $\beta_N + 1 < 0$, there exists a constant $C_{J,N,i,j} > 0$ such that for any $|x| \geq 1$ and any $t \in J$

$$\Big|\, \partial_t^i \partial_x^j \Big(f(t,x) - \Big(a_*^\pm(t)\log(\pm x) + \sum_{k=0}^N a_k^\pm(t)(\pm x)^{\beta_k+1} \Big)\Big) \Big| \leq C_{J,N,i,j}|x|^{(\beta_{N+1}+1)-j}. \tag{29}$$

Proof of Lemma 3.1. If $f \in C^\infty(\mathbb{R}\times\mathbb{R})$ has an asymptotic expansion as in (28), then clearly, $f_x \in \mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$, hence $f \in \mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$. Let us prove the converse statement. As the asymptotic expansions for $x \to +\infty$ and $x \to -\infty$ of an element $f \in \mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ are obtained in a similar way let us consider the case $x \to +\infty$ only. First we treat the case where $\beta + 1 \geq 0$. By definition, for an element $f \in \mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$, $f_x \in \mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ and hence has an asymptotic expansion

$$f_x \sim \sum_{k=0}^\infty b_k^+(t)\,x^{\beta_k} \quad \text{as} \quad x \to \infty \tag{30}$$

where $\beta = \beta_0 > \beta_1 > \ldots$ with $\lim_{k\to\infty} \beta_k = \infty$. Without loss of generality we assume that $\beta_m = -1$ for some $m \geq 0$.[1] Formally, the claimed result is obtained by integrating term by term the right hand side of (30) with respect to the x-variable. In order to make this argument rigorous we argue as follows: For any $N \geq m+1$ and $x \in \mathbb{R}$ consider the quantity

$$Q_N(t,x) := -\chi_+(x) \int_x^\infty \Big(f_x(t,s) - \sum_{k=0}^N b_k^+(t) s^{\beta_k} \Big)\, ds \tag{31}$$

[1] Take $b_m^+(t) \equiv 0$ if necessary.

where $\chi_+(x)$ is a smooth cut-off function with $\chi_+(x) = 0$ for $x \leq 1/2$ and $\chi_+(x) = 1$ for $x \geq 1$. As $f_x \in \mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ and $\beta_{m+1} < -1$ it follows that the improper integral in (31) exists and if $x \geq 1$, $\partial_x Q_N(t,x) = f_x(t,x) - \sum_{k=0}^N b_k^+(t) x^{\beta_k}$. Hence, $\partial_x Q_N$ is in $\mathcal{S}_{\beta_{N+1}}(\mathbb{R}\times\mathbb{R})$. We claim that Q_N is in $\mathcal{S}_{\beta_{N+1}+1}(\mathbb{R}\times\mathbb{R})$. To show this it remains to estimate $\partial_t^i Q_N(t,x)$. It follows from (30) that for any compact interval $J \subseteq \mathbb{R}$, $i \geq 0$, and $N \geq m+1$, there exists a constant $C_{J,N,i} > 0$ such that for any $x \geq 1$, $t \in J$

$$\begin{aligned} |\partial_t^i Q_N(t,x)| &\leq \int_x^\infty |\partial_t^i (f_x(t,s) - \sum_{k=0}^N b_k^+(t) s^{\beta_k})| \, ds \\ &\leq C_{J,N,i} \int_x^\infty s^{\beta_{N+1}} \, ds \\ &\leq C_{J,N,i} \, \frac{x^{\beta_{N+1}+1}}{|\beta_{N+1}+1|} \, . \end{aligned} \tag{32}$$

Computing the integral in (31) one gets for $x \geq 1$

$$Q_N(t,x) = f(t,x) - \Big(c^+(t) + b_m^+(t) \log x + \sum_{0\leq k\leq N, k\neq m} \frac{b_k^+(t)}{\beta_k+1} \, x^{\beta_k+1} \Big) \tag{33}$$

where

$$c^+(t) := f(t,1) - \sum_{k=0}^{m-1} \frac{b_k^+(t)}{\beta_k+1} + \int_1^\infty \Big(f_x(t,s) - \sum_{k=0}^m b_k^+(t) s^{\beta_k} \Big) \, ds \, . \tag{34}$$

(Note that the integral in (34) converges as the integrand is estimated locally uniformly in t by $O(s^{\beta_{m+1}})$ with $\beta_{m+1} < -1$.) The desired estimate (29) of $f(t,x)$ for $x \to +\infty$ follows from (30), (32), and (33).

The case $\beta + 1 < 0$ is treated in a similar way, actually it is easier than the case $\beta + 1 \geq 0$. □

Proof of Theorem 1.1. We will show that the claimed results follow from Theorem 2.5 and Lemma 3.2 stated below, combined with results in [3, 2] - see Appendix B for a summary of these results. Indeed, for a given $r_0 \in \mathcal{S}_\beta(\mathbb{R})$, the Miura image $q_0 := r_{0x} + r_0^2$ belongs to $\mathcal{S}_\delta(\mathbb{R})$ with $\delta := \max\{2\beta, \beta - 1\} < 1$. According to the results in [3, 2] (cf. Theorem B.1, B.2 in Appendix B) there exists a unique solution $q \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$ of the KdV equation (3) with initial data $q|_{t=0} = q_0$. As $\delta < 1$ the solution $q = q(t,x)$ satisfies the growth condition (16). In particular, according to Proposition 2.2 the linear initial value problem

$$\psi_t(t,x) = 2q(t,x)\psi_x(t,x) - q_x(t,x)\psi(t,x) \tag{35}$$

$$\psi|_{t=0} = \psi_0(x) := e^{\int_0^x r_0(s)\, ds} \tag{36}$$

has a unique (within the class of C^∞-functions) globally (in time) defined positive solution, $\psi = \psi(t,x) > 0$, which satisfies $-\psi_{xx} + q\psi = 0 \ \forall t, x \in \mathbb{R}$. It follows from item ($a$) of Theorem 2.5 that the function $r(t,x) = \psi_x(t,x)/\psi(t,x)$ is a solution of (1)-(2). It is easy to see that the function

$$p = p(t,x) := \log \psi(t,x)$$

satisfies

$$p_t(t,x) = 2q(t,x)p_x(t,x) - q_x(t,x) \tag{37}$$

$$p|_{t=0} = p_0(x) \tag{38}$$

with initial data $p_0(x) = \int_0^x r_0(s)\, ds \in \mathcal{S}^*_{\beta+1}(\mathbb{R})$. According to Lemma 3.2 below the function $p(t,x)$ belongs to $\mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ and therefore $r = \partial_x p \in \mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$.

The uniqueness of the solution $r = r(t,x)$ in the class $\mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ follows from the uniqueness of the solution $q = q(t,x)$ in the class $\mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$ (cf. Theorem B.2 in Appendix B) and Theorem 2.5 (b). □

The proof of Theorem 1.1 used the following lemma.

Lemma 3.2. *Let $\beta < 1/2$ and $\delta := \max\{2\beta, \beta-1\} < 1$. Consider the initial value problem (37)-(38) where $q \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$. Then for any initial data $p_0 \in \mathcal{S}^*_{\beta+1}(\mathbb{R})$ there exists a solution $p \in \mathcal{S}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ of (37)-(38). This solution is unique in $C^\infty(\mathbb{R}\times\mathbb{R})$.*

In order to prove Lemma 3.2 we will first construct formal series $\chi_\pm(t,x)$ having the form (28) and satisfying the evolution equation (37)-(38) formally for $x \to \pm\infty$. As the cases $x \to +\infty$ and $x \to -\infty$ are treated in the same way we restrict our attention only to the case $x \to +\infty$.

Let $p_0 \in \mathcal{S}^*_{\beta+1}(\mathbb{R})$ be the initial data in (37)-(38). By Lemma 3.1, $p_0(x)$ has an asymptotic expansion for $x \to +\infty$ of the form

$$p_0(x) \sim \begin{cases} \sum_{k=0}^\infty p_k^+ x^{\beta_k+1} + p_*^+ \log x & \text{if} \quad \beta+1 \geq 0 \\ c^+ + \sum_{k=0}^\infty p_k^+ x^{\beta_k+1} & \text{if} \quad \beta+1 < 0 \end{cases}$$

where $\beta_0 := \beta < 1/2$ and $\beta_0 > \beta_1 > \ldots$, $\lim_{k\to\infty} \beta_k = -\infty$. As a solution p of (37)-(38) gives rise to the 1-parameter family of solutions $p + \texttt{const}$, we can assume without loss of generality that the constant c^+ in the asymptotic expansion for $p_0(x)$ vanishes,

$$p_0(x) \sim \sum_{k=0}^\infty p_k^+ x^{\beta_k+1} + p_*^+ \log x \quad \text{as} \quad x \to \infty\,. \tag{39}$$

Here $\beta < 1/2$ but not necessarily $\beta + 1 \geq 0$. By assumption, $q \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$ and hence it has an asymptotic expansion for $x \to +\infty$ of the form

$$q(t,x) \sim \sum_{k=0}^\infty c_k^+(t) x^{\delta_k} \tag{40}$$

where $\delta_0 = \delta := \max\{2\beta, \beta - 1\} < 1$ and $\delta_0 > \delta_1 > \ldots$, $\lim_{k\to\infty} \delta_k = -\infty$. Consider the set

$$\Delta := \{\delta_k\}_{k\geq 0}.$$

In order to find a *formal* solution $\chi_+(t,x)$ of (37)-(38) we will have to extend the set of exponents $\{\beta_k + 1\}_{k\geq 0}$ appearing in (39) to a larger discrete set $\bar{B}$ with the same upper limit as $\{\beta_k + 1\}_{k\geq 0}$ so that the exponents appearing in the asymptotic expansions of the left and right hand side of (37) belong to $\bar{B}$. To construct $\bar{B}$ we first need to extend the set Δ.

Lemma 3.3. *There exists an unbounded discrete set $\bar{\Delta} \subseteq \mathbb{R}$ with $\Delta \subseteq \bar{\Delta}$ such that*

(a) $\max \bar{\Delta} = \max \Delta = \delta < 1$*;*

(b) if $\delta', \delta'' \in \bar{\Delta}$ then $\delta' + \delta'' - 1 \in \bar{\Delta}$;

(c) if $\delta' \in \bar{\Delta}$ then $\delta' - 1 \in \bar{\Delta}$.

Proof. First note that a set $\bar{\Delta}$ satisfies (b) iff $\bar{\Delta} - 1 := \{\delta' - 1 \,|\, \delta' \in \bar{\Delta}\}$ satisfies

$$\delta', \delta'' \in \bar{\Delta} - 1 \quad \text{implies} \quad \delta' + \delta'' \in \bar{\Delta} - 1\,. \tag{41}$$

It is easy to see that the set $\Delta_1 \subseteq \mathbb{R}$,

$$\Delta_1 := \Big\{ \sum_{i\in J} \delta_i \,|\, \delta_i \in \Delta - 1, J \subseteq \mathbb{Z}_{\geq 0} \text{ is finite and } J \neq \emptyset \Big\}$$

is discrete, satisfies (41) and that $\max \Delta_1 = \delta - 1$. Consider the set

$$\bar{\Delta}_1 := \{\delta' - k \,|\, \delta' \in \Delta_1, k \in \mathbb{Z}_{\geq 0}\}.$$

Then

$$\delta', \delta'' \in \bar{\Delta}_1 \quad \text{implies} \quad \delta' + \delta'' \in \bar{\Delta}_1;$$

in addition, $\bar{\Delta}_1$ is unbounded and discrete. Moreover $\max \bar{\Delta}_1 = \delta - 1$, and $\delta' - 1 \in \bar{\Delta}_1$ for any $\delta' \in \bar{\Delta}_1$. Hence, the set $\bar{\Delta} := \bar{\Delta}_1 + 1$ satisfies claims (a)-(c) of the lemma. □

We extend in the sum in (40) the set of exponents Δ to $\bar{\Delta}$ by setting the new coefficients in (40) all equal to zero. Hence without loss of generality, one can – and in the sequel we will – assume that the set of the exponents $\Delta = \{\delta_k\}_{k\geq 0}$ in (40) satisfies conditions (a)-(c) of Lemma 3.3.

Let us now introduce the following subsets of $\mathbb{R}$,

$$B := \{\beta_k\}_{k\geq 0}$$

and

$$\bar{B} := \{\beta' + \delta' \,|\, \beta' \in B,\ \delta' \in \Delta\} \cup \Delta \cup \{\beta' + 1 \,|\, \beta' \in B\} \tag{42}$$

Lemma 3.4. *The set $\bar{B}$ is discrete and has the following properties:*

(i) $\max \bar{B} = \beta + 1$;

(ii) *if* $\delta' \in \Delta$ *and* $\beta' \in \bar{B}$, *then* $\delta' + \beta' - 1 \in \bar{B}$;

(iii) *the set* $\{\delta' - 1 \mid \delta' \in \Delta\}$ *is contained in* $\bar{B}$.

Proof of Lemma 3.4. The proof that $\bar{B}$ is discrete follows from the arguments used in the proof of Lemma 3.3.
(i) As $\beta < 1/2$ and $\delta = \max\{2\beta, \beta - 1\} < 1$ one gets $\max \bar{B} = \max\{\delta, \beta + \delta, \beta + 1\} = \beta + 1$.
(ii) follows from the fact that Δ has property Lemma 3.3 (*b*). Indeed, as any $\beta' \in \bar{B}$ can be written in the form $\beta' = \beta'' + \delta''$ ($\beta'' \in B$, $\delta'' \in \Delta$), $\beta' = \delta''$, or $\beta' = \beta'' + 1$ and as by Lemma 3.3(b), for any $\delta' \in \Delta$, one has $\delta''' := \delta' + \delta'' - 1 \in \Delta$, it follows that

$$\delta' + \beta' - 1 = \begin{cases} \delta' + (\beta'' + \delta'') - 1 = \delta''' + \beta'' \in \bar{B} \\ \delta' + \delta'' - 1 \in \Delta \subseteq \bar{B} \\ \delta' + (\beta'' + 1) - 1 = \delta' + \beta'' \in \bar{B} \end{cases}$$

(iii) It follows from statement (c) of Lemma 3.3 that for any $\delta' \in \Delta$, one has $\delta' - 1 \in \Delta$ and as $\Delta \subseteq \bar{B}$, (*c*) then follows. □

Proof of Lemma 3.2. First we prove that for any

$$q(t, x) \sim \sum_{k=0}^{\infty} c_k^{\pm}(t)(\pm x)^{\delta_k} \quad \text{as} \quad x \to \pm\infty \tag{43}$$

with exponents $\Delta = \{\delta_k\}_{k \geq 0}$, $\delta_0 = \delta > \delta_1 > \ldots$, satisfying claims (a)-(b) of Lemma 3.3, the initial value problem (37)-(38) with $p_0(x)$ satisfying (39) has a formal solution $\chi_+(t, x)$ given by ($t \in \mathbb{R}, x > 0$)

$$\chi_+(t, x) = \sum_{k=0}^{\infty} a_k^+(t) x^{\bar{\beta}_k + 1} + a_*^+(t) \log x \tag{44}$$

where $\{\bar{\beta}_k + 1\}_{k \geq 0} = \bar{B}$ with $\bar{\beta}_0 > \bar{\beta}_1 > \ldots$. The existence of a formal solution $\chi_-(t, x)$ for $t \in \mathbb{R}$, $x < 0$ follows by the same arguments. Let us stress that the exponents $\beta_k + 1$ in the asymptotic expansion of the initial data $p_0(x)$ (cf. (39)) belong to the set $\{\beta' + 1 \,|\, \beta' \in B\}$ which is included in the larger set $\bar{B}$. Hence, the coefficients $a_k^+(0)$ in the asymptotic expansion (44) evaluated at $t = 0$ are zero or coincide with some of the constants p_k^+. Moreover $a_*^+(0) = p_*^+$.

Substituting (43) and (44) into (37) and using the notation $\dot{} = \frac{d}{dt}$ one obtains, in the case $x > 0$,

$$\begin{aligned} \dot{a}_*^+(t) \log x + \dot{a}_0^+(t) x^{\beta+1} + \dot{a}_1^+(t) x^{\bar{\beta}_1+1} + \dot{a}_2^+(t) x^{\bar{\beta}_2+1} + \ldots = \\ = 2\Big(c_0^+(t) x^{\delta} + c_1^+(t) x^{\delta_1} + \ldots\Big)\Big(a_*^+(t) x^{-1} + \\ (\beta+1) a_0^+(t) x^{\beta} + (\bar{\beta}_1 + 1) a_1^+(t) x^{\bar{\beta}_1} + \ldots\Big) \\ -\Big(\delta c_0^+(t) x^{\delta-1} + \delta_1 c_1^+(t) x^{\delta_1 - 1} + \ldots\Big) \end{aligned} \tag{45}$$

The maximal power of x on the right side of (45) is not bigger than $m_r = \max\{\beta+\delta, \delta-1\}$. As $\beta < 1/2$ and $\delta = \max\{2\beta, \beta-1\} < 1$ one obtains that $\beta + 1 > m_r$. Hence, $\dot{a}_0^+(t) = 0$ and thus $a_0^+(t) = a_0^+(0)$. Comparing the coefficients in (45) we also obtain that $\dot{a}_*^+(t) = 0$ and hence $a_*^+(t) = a_*^+(0)$.

Comparing the coefficients in (45) corresponding to terms of order $\bar{\beta}_k + 1$ in x one obtains that for any $k \geq 1$

$$\dot{a}_k^+(t) = P_k^+(a_0^+, a_1^+(t), ..., a_{k-1}^+(t)) + F_k^+(t) \tag{46}$$

where P_k^+ is a linear combination of the variables $a_0^+, ..., a_{k-1}^+$ with coefficients which are smooth functions of $t \in \mathbb{R}$. The term $F_k^+(t)$ is equal to $2c_{i_k}^+(t)p_*^+ - \delta_{i_k} c_{i_k}^+(t)$ iff there exists an index $i_k \geq 0$ such that $\bar{\beta}_k + 1 = \delta_{i_k} - 1$. If there is no such i_k then $F_k^+(t) \equiv 0$. Let us prove formula (46). It is clear from (45) that the right side of (46) is a sum of a linear polynomial of the variables $a_0^+, a_1^+, ...$ and an inhomogeneous term $F_k^+(t)$ of the form described above. Assume that there exists a_n^+, $n \geq k$, that enters as a linear term on the right side of (46). Then clearly there exists $m_n \geq 0$ such that

$$\bar{\beta}_n + \delta_{m_n} = \bar{\beta}_k + 1\ .$$

As $\bar{\beta}_n \leq \bar{\beta}_k$ and $\delta_{m_n} \leq \delta < 1$, it follows that $\bar{\beta}_n + \delta_{m_n} < \bar{\beta}_k + 1$. This contradiction proves (46).

Integrating equation (46) for $k = 1, 2, ...$ we find recursively the coefficients $a_k^+(t)$ in terms of the initial values $(a_i^+(0))_{0 \leq i \leq k}$. Clearly, the formal solution $\chi_+(t, x)$ satisfies (45) and by construction $\chi_+(0, x) = p_0(x)$. Arguing similarly we find a formal solution $\chi_-(t, x)$ for $t \in \mathbb{R}$, $x < 0$.

Next we show how the constructed formal solutions

$$\chi_\pm(t, x) = \sum_{k=0}^{\infty} a_k^\pm(t)(\pm x)^{\bar{\beta}_k + 1} + a_*^\pm(t)\log(\pm x)$$

lead to a solution of (37)-(38). Choose $f(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R})$ so that f has asymptotic expansions of the form

$$f(t, x) \sim \sum_{k=0}^{\infty} a_k^\pm(t)(\pm x)^{\bar{\beta}_k + 1} + a_*^\pm(t)\log(\pm x) \ \text{ as } \ x \to \pm\infty \tag{47}$$

with coefficients $(a_k^\pm(t))_{k \geq 0}$, $a_*^\pm(t)$ defined as above. The existence of such a function f follows, for example, from [16, Proposition 3.5]. Following [3, 2] we will call the function $f(t, x)$ an *asymptotic solution* of (37)-(38). Let $f_0 := f|_{t=0}$.

With the help of the asymptotic solution f we want to find a solution $p(t, x)$ of (37)-(38) of the form

$$p(t, x) := f(t, x) + s(t, x) \tag{48}$$

where $s(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R})$ has to be determined so that s and all derivatives $\partial_t^l \partial_x^k s$ are fast decaying as $|x| \to \infty$. Substituting (48) into (37)-(38) one obtains the linear evolution equation

$$s_t(t,x) = 2q(t,x)s_x(t,x) + \eta(t,x) \tag{49}$$
$$s|_{t=0} = s_0(x) \tag{50}$$

where $\eta(t,x) := -f_t(t,x) + 2q(t,x)f_x(t,x) - q_x(t,x)$ belongs to $\mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ (as $f(t,x)$ is an asymptotic solution of (37)-(38)) and $s|_{t=0} = p_0(x) - f_0(x) \in \mathcal{S}(\mathbb{R})$, where as usual, $\mathcal{S}(\mathbb{R})$ denotes the functions of Schwartz class. By definition, $g \in C^\infty(\mathbb{R}\times\mathbb{R})$ belongs to the space $\mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ iff for any compact interval $J \subseteq \mathbb{R}$ and any $k,i,j \geq 0$ there exists a constant $C_{J,k,i,j} > 0$ such that for any $|x| \geq 1$ and $t \in J$

$$|\partial_t^i \partial_x^j g(t,x)| \leq C_{J,k,i,j}|x|^{-k}.$$

In particular, if $g \in \mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ then for any given $t \in \mathbb{R}$, the function $g(t,\cdot)$ belongs to $\mathcal{S}(\mathbb{R})$.

Due to Lemma A.3, we can find a solution $s \in \mathcal{S}_{-\infty}(\mathbb{R}\times\mathbb{R})$ of (49)-(50) which proves the existence part of Lemma 3.2. The uniqueness of the solution $p(t,x)$ in $C^\infty(\mathbb{R}\times\mathbb{R})$ follows from Lemma A.1. □

4 Proof of Theorem 1.2 and Theorem 1.3

In this section we prove the global (in time) existence and the uniqueness of solutions of the mKdV equation stated in Theorem 1.2 and Theorem 1.3.

Before proving these theorems we introduce the following auxiliary spaces

$$\mathcal{O}^*_{\beta+1}(\mathbb{R}\times\mathbb{R}) := \{f \in C^\infty(\mathbb{R}\times\mathbb{R}) \,|\, f_x \in \mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})\}$$

and

$$o^*_{\beta+1}(\mathbb{R}\times\mathbb{R}) := \{f \in C^\infty(\mathbb{R}\times\mathbb{R}) \,|\, f_x \in o_\beta(\mathbb{R}\times\mathbb{R})\}\,.$$

Proof of Theorem 1.2. We follow the arguments in the proof of Theorem 1.1. Given $r_0 \in \mathcal{O}_\beta(\mathbb{R})$ we get $q_0 := r_0' + r_0^2$ which belongs to the space $\mathcal{O}_\delta(\mathbb{R})$ with $\delta := \max\{2\beta, \beta-1\} < 1$. By Theorem 2 in [5] (cf. Theorem B.3, Appendix B) there exists a solution $q \in \mathcal{O}_\delta(\mathbb{R}\times\mathbb{R})$ of the KdV initial value problem

$$q_t - 6qq_t + q_{xxx} = 0, \quad q|_{t=0} = q_0\,.$$

As $\delta < 1$ the solution $q(t,x)$ satisfies the growth condition (16). Let $\psi(t,x) > 0$ be the globally defined unique solution of (35)-(36) (see also Proposition 2.2). According to Theorem 2.5 (a) the function $r(t,x) = \psi_x(t,x)/\psi(t,x)$ is a solution of the mKdV initial value problem (1)-(2). Then $p(t,x) := \log\psi(t,x)$ satisfies (37)-(38) with $p_0(x) = \int_0^x r_0(s)\,ds$. As p_0 is in $\mathcal{O}^*_{\beta+1}(\mathbb{R})$ the solution $p(t,x)$ of (37)-(38) belongs to $\mathcal{O}^*_{\beta+1}(\mathbb{R}\times\mathbb{R})$ (cf. Lemma 4.1 below). In particular $r(t,x) = p_x(t,x) \in \mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})$.

The uniqueness of the solution $r(t,x)$ constructed above follows from Theorem 2.5 (b) and the uniqueness result for KdV in Theorem 1 in [4] (cf. Theorem B.2, Appendix B). □

In the proof of Theorem 1.2 we used the following analogue of Lemma 3.2.

Lemma 4.1. *Let $\beta < 1/2$ and $\delta := \max\{2\beta, \beta - 1\} < 1$. Consider the initial value problem* (37)-(38) *where $q \in \mathcal{O}_\delta(\mathbb{R} \times \mathbb{R})$. Then for any initial data $p_0 \in \mathcal{O}^*_{\beta+1}(\mathbb{R})$ there exists a solution $p \in \mathcal{O}^*_{\beta+1}(\mathbb{R} \times \mathbb{R})$ of* (37)-(38) *which is unique in $C^\infty(\mathbb{R} \times \mathbb{R})$.*

Proof of Lemma 4.1. The lemma is proved by the same arguments as the ones used in the proof of Lemma A.3 (see also [5], Proposition 1). □

Proof of Theorem 1.3. The proof is similar to the proof of Theorem 1.2 and is based on the existence and uniqueness results for the initial value problem of KdV of [4, 5] (cf. Theorem B.2, B.4 in Appendix B) and on a variant of Lemma 4.1 where the spaces $\mathcal{O}_\beta$ and $\mathcal{O}^*_{\beta+1}$ are replaced by the spaces o_β and $o^*_{\beta+1}$ respectively. □

We conclude this section by stating a more general uniqueness result for the mKdV initial value problem (1)-(2). Let $I = (a, b) \subseteq \mathbb{R}$ with $-\infty \le a < b \le \infty$. Denote by $\mathcal{G}(\mathbb{R} \times \mathbb{R})$ the linear space of functions $r(t, x)$ in $C^\infty(\mathbb{R} \times \mathbb{R})$ such that for any compact interval $J \subseteq I$ one has for $|x| \ge 1$ and any $k \ge 1$

$$r(t, x) = o(\sqrt{|x|}) \quad \text{and} \quad \partial_x^k r(t, x) = O(1/\sqrt{|x|}),$$

uniformly in $t \in J$. The following theorem follows in a straightforward way from Theorem 2.5 (b) and Theorem 1 in [4] (cf. Theorem B.2, Appendix B).

Theorem 4.2. *There exists at most one solution of the mKdV initial value problem (1)-(2) in $\mathcal{G}(\mathbb{R} \times \mathbb{R})$.*

5 Spectral invariance

In this section we prove the spectral invariance of the Schrödinger operator $L(t) = -\partial_x^2 + q(t, x)$ under the KdV flow (see Theorem 1.6 stated in the introduction) and the spectral invariance of the imedance operator $T(t) = -\frac{d^2}{dx^2} - 2r(t, x)\frac{d}{dx}$ under the mKdV flow (Theorem 5.3 below).

Invariance of the spectrum: We prove the two statements of Theorem 1.6 separately using two different methods. The first statement follows from the second one in case $\beta < 1$, but we still provide a separate proof, because it is elementary and self-contained, whereas the proof of the second statement relies on a result from [4].

Let $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \le 1$ and $I = (a, b) \subseteq \mathbb{R}$, $-\infty \le a < b \le +\infty$. Consider the t-parameter family of self-adjoint operators in $L^2(\mathbb{R})$

$$L(t) := -\partial_x^2 + q(t, x)\,.$$

As for any $t \in I$ there exists a constant $C = C(t) > 0$ such that $|q(t, x)| \le C\,|x|$ for $|x| \ge 1$, by Sears theorem (cf. [1, Chapter II]) the symmetric operator $u \mapsto -u'' + q(t)u$ in $L^2(\mathbb{R})$ with domain $C_0^\infty(\mathbb{R})$ is essentially self-adjoint. We define $L(t)$ to be the closure of this operator.

In order to prove Theorem 1.6 we first extend the Q-L formalism for operators with spectral parameter. Let $q \in C^\infty(I \times \mathbb{R})$. For a given $\lambda \in \mathbb{R}$ consider the operators

$$Q_\lambda(t) := (4\lambda + 2q(t,x))\partial_x - q_x(t,x)$$

and

$$L_\lambda(t) := L(t) - \lambda\,.$$

The operators Q_λ and L_λ satisfy the commutator relation

$$\dot{L}_\lambda = [Q_\lambda, L_\lambda] + 4q_x L_\lambda + KdV(q)\,. \tag{51}$$

In particular, $q(t,x)$ is a solution of the KdV equation, $KdV(q) = 0$, if and only if

$$\dot{L}_\lambda = [Q_\lambda, L_\lambda] + 4q_x L_\lambda \tag{52}$$

for some $\lambda \in \mathbb{R}$ (hence, for all $\lambda \in \mathbb{R}$). Now, assume that $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \leq 1$ is a solution of the KdV equation and let $\psi \in C^\infty(I \times \mathbb{R})$ be the solution of the first order evolution equation

$$\begin{aligned} \psi_t(t,x) &= (4\lambda + 2q(t,x))\psi_x(t,x) - q_x(t,x)\psi(t,x) & (53)\\ \psi(t',x) &= \psi_0(x)\,. & (54) \end{aligned}$$

Let $\varphi(t,x) := L_\lambda(t)\psi(t,x)$. Using (52) we get

$$\begin{aligned} \varphi_t &= \dot{L}_\lambda\psi + L_\lambda\psi_t \\ &= ([Q_\lambda, L_\lambda] + 4q_x L_\lambda)\psi + L_\lambda Q_\lambda \psi \\ &= Q_\lambda(L_\lambda\psi) + 4q_x(L_\lambda\psi) \\ &= (4\lambda + 2q)\varphi_x + 3q_x\varphi\,. \end{aligned}$$

Hence $\varphi \equiv \varphi(t,x)$ is a solution of the initial value problem (cf. Lemma A.1)

$$\begin{aligned} \varphi_t(t,x) &= (4\lambda + 2q(t,x))\varphi_x(t,x) + 3q_x(t,x)\varphi(t,x) \\ \varphi|_{t=t'} &= L_\lambda(t')\psi_0\,. \end{aligned}$$

Lemma 5.1. *Let $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \leq 1$ be a solution of the KdV equation and let J be a compact interval in I. Then there exists a constant $C(J) > 0$ such that for any $t, t' \in J$ and for any $\psi_0 \in C_0^\infty(\mathbb{R})$ the solution ψ of the evolution equation* (53)-(54) *satisfies the inequalities*

$$\|\psi(t)\| \leq C(J)\,\|\psi_0\| \tag{55}$$

and

$$\|L_\lambda(t)\psi(t)\| \leq C(J)\,\|L_\lambda(t')\psi_0\|\,. \tag{56}$$

Proof of Lemma 5.1. Let $\xi(t; t', x_0)$ be the solution of the ordinary differential equation

$$\dot{\xi} = -(4\lambda + 2q(t,\xi)) \tag{57}$$

$$\xi|_{t=t'} = x\,. \tag{58}$$

It follows from the equation of variation of (57)-(58) that

$$\xi_x(t;t',x) = e^{-2\int_{t'}^{t} q_x(\tau,\xi(\tau;t',x))\,d\tau}\,. \tag{59}$$

By the method of characteristics we get that for any $t'' \in I$,

$$\phi(t'',\xi(t'';t',x)) = \phi_0(x)\,e^{3\int_{t'}^{t''} q_x(\tau,\xi(\tau;t',x))\,d\tau}\,,$$

or, equivalently,

$$\phi(t'',y) = \phi_0(\xi(t';t'',y))\,e^{3\int_{t'}^{t''} q_x(\tau,\xi(\tau;t'',y))\,d\tau}\,. \tag{60}$$

Assuming that $t', t'' \in J$ for some compact interval J in I we obtain from (59), (60), and $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \le 1$ that there exist constants $C(J) > 0$ and $C_1(J) > 0$ such that for any $\phi_0 \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}
\|\phi(t'')\|^2 &= \int_{-\infty}^{\infty} \phi_0(\xi(t';t'',y))^2\,e^{6\int_{t'}^{t''} q_x(\tau,\xi(\tau;t'',y))\,d\tau}\,dy \\
&\le C_1(J)\int_{-\infty}^{\infty} \phi_0(\xi(t';t'',y))^2\,dy \\
&= C_1(J)\int_{-\infty}^{\infty} \phi_0(x)^2|\xi_x(t'';t',x)|\,dx \\
&\le C(J)^2\,\|\phi_0\|^2\,.
\end{aligned}$$

Arguing as above and choosing $C(J) > 0$ larger if necessary one proves that for any $t'' \in J$ and for any $\psi_0 \in C_0^\infty(\mathbb{R})$ the solution $\psi(t,x)$ of (53)-(54) satisfies

$$\|\psi(t'')\| \le C(J)\,\|\psi_0\|\,.$$

This completes the proof of the lemma. □

Proof of the first statement of Theorem 1.6. A point $\lambda \subseteq \mathbb{R}$ belongs to the spectrum of the self-adjoint operator $L(t)$ iff there exist a sequence $(\varepsilon_k)_{k\ge1}$ of positive numbers $\varepsilon_k > 0$ with $\lim\limits_{k\to\infty} \varepsilon_k = 0$ and a sequence of functions $(\psi_k)_{k\ge1} \subseteq C_0^\infty(\mathbb{R})$, $\psi_k \not\equiv 0$, such that for any $k \ge 1$,

$$\|L_\lambda(t)\psi_k\| \le \varepsilon_k\|\psi_k\|\,. \tag{61}$$

Assume that $\lambda \in \mathtt{spec}\,L(t')$. We will prove that for any $t'' \in I$, $\lambda \in \mathtt{spec}\,L(t'')$. Take (ϵ_k) and (ψ_k) as above so that (61) is satisfied with $t = t'$. Using Lemma 5.1 we get the following estimates for the solution $\psi_k(t,x)$ of equation (53) with initial data $\psi_k|_{t=t'} = \psi_k$

$$\begin{aligned}
\|L(t'')\psi_k(t'')\| &\le C(J)\,\|L(t')\psi_k\| \\
&\le \epsilon_k C(J)\,\|\psi_k\| \le \epsilon_k C(J)^2\|\psi_k(t'')\|\,.
\end{aligned} \tag{62}$$

As $\epsilon_k C(J)^2 \to 0$ as $k \to \infty$ we get from (62) that $\lambda \in \mathtt{spec}\,L(t'')$. As the inclusion $\mathtt{spec}\,L(t') \subseteq \mathtt{spec}\,L(t'')$ was proved for any $t', t'' \in I$, $\mathtt{spec}\,L(t') = \mathtt{spec}\,L(t'')$.

Any eigenfunction of $L(t')$ with eigenvalue $\lambda \in \mathbb{R}$ coincides up to a set of measure zero with a smooth solution of the differential equation $-\psi_0''(x) + q(t,x)\psi_0(x) = \lambda\psi_0(x)$ such that $\psi_0 \in L^2(\mathbb{R})$. Let ψ be the solution of the evolution equation (53)-(54). Arguing as in the proof of Lemma 5.1 one sees that the inequalities (55) and (56) still hold. In particular, we get that for any $t \in I$, $\psi(t) \in L^2(\mathbb{R})$ and $L(t)\psi(t) = \lambda\psi(t)$. The coincidence of the multiplicities follows from the uniqueness of solution for the initial value problem (53)-(54). □

Remark 5.2. *In addition to the commutator relation* (51)*, the operators* L_λ *and* Q_λ *satisfy for arbitrary* $\lambda \in \mathbb{R}$ *the identity*

$$L_\lambda(\partial_t - Q_\lambda) + [L_\lambda(\partial_t - Q_\lambda)]^* = -KdV(q)$$

where P^* *denotes the formal adjoint of a differential operator* P *in* $L^2(I \times \mathbb{R})$*.*

Proof of the unitary equivalence in Theorem 1.6. Assume either $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta < 1$ or $q \in o_\beta(I \times \mathbb{R})$ with $\beta \leq 1$ where $I = (a,b) \subseteq \mathbb{R}$ and $-\infty \leq a < b \leq +\infty$. In addition, assume for simplicity that $0 \in I$. Following Lax [12] we consider the t-parameter family of third order linear differential operators

$$A(t) := -4\partial_x^3 + 6q(t,x)\partial_x + 3q_x(t,x)\ . \tag{63}$$

The operators $L(t) = -\partial_x^2 + q(t,x)$ and $A(t)$ satisfy the commutator relation

$$\dot{L} = [A, L] + KdV(q)$$

where $KdV(q) = q_t - 6qq_x + q_{xxx}$. Assuming that $q(t,x)$ is a solution of the KdV equation we obtain that $L(t)$ and $A(t)$ satisfy the classical Lax pair relation

$$\dot{L} = [A, L]\,. \tag{64}$$

Consider the linear evolution equation

$$\psi_t(t) = A(t)\psi(t) \tag{65}$$

$$\psi|_{t=0} = \psi_0 \tag{66}$$

with initial data ψ_0 in the Schwartz space $\mathcal{S}(\mathbb{R})$. The existence of a solution of (65)-(66) evolving in $\mathcal{S}(\mathbb{R})$ on the whole time interval (a,b) was solved positively in [4] by applying a difference scheme method as in [2, 14]. According to [4, Theorem 2], (65)-(66) has a solution $\psi \in C^1(I, \mathcal{S}(\mathbb{R}))$.[2] By applying the Holmgren's principle one easily sees that the solution ψ is indeed unique. Denote by $\Psi(t)$ the operator

$$\Psi(t) : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}), \quad \psi_0 \mapsto \psi(t). \tag{67}$$

[2]In fact, this solution lies in $C^\infty(I, \mathcal{S}(\mathbb{R}))$.

For any $\psi_0 \in \mathcal{S}(\mathbb{R})$ one gets by integration by parts

$$\frac{d}{dt}\Big(\Psi(t)\psi_0, \Psi(t)\psi_0\Big) = \Big(A(t)\Psi(t)\psi_0, \Psi(t)\psi_0\Big) + \Big(\Psi(t)\psi_0, A(t)\Psi(t)\psi_0\Big) = 0$$

where $(\cdot,\cdot)$ denotes the L^2 scalar product. Hence, the operator (67) preserves the L^2 norm.

Let $\psi(t)$ be the solution of (65)-(66). By (64) and the Leibniz rule we get

$$(L\psi)^{\cdot} = \dot{L}\psi + L\dot{\psi} = [A, L]\psi + LA\psi = A(L\psi)\,. \tag{68}$$

Hence, $L(t)\psi(t)$ is a solution of (65) with initial data $L(0)\psi_0$. The latter result together with the uniqueness of the solution of (65) with the initial data $L(0)\psi_0$ imply that

$$\Psi(t)L(0)\psi_0 = L(t)\Psi(t)\psi_0 \quad \forall\, \psi_0 \in \mathcal{S}(\mathbb{R})\,. \tag{69}$$

The claimed unitary equivalence of the operators $L(0)$ and $L(t)$ then follows easily from (69) and the Sears theorem. $\square$

The impedance operator: Here we prove that the spectrum of the impedance operator

$$T(t) := -\frac{d^2}{dx^2} - 2r(t,x)\,\frac{d}{dx}$$

where $r(t,x)$ is a solution of the mKdV equation in $\mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \leq 1/2$ and $I = (a,b)$, $-\infty \leq a < b \leq +\infty$, is invariant. For any given $t \in (a,b)$ the operator $-\frac{d^2}{dx^2} - 2r(t,x)\,\frac{d}{dx}$ with domain $C_0^\infty(\mathbb{R})$ is essentially self-adjoint in the weighted L^2-space $L^2(\mathbb{R}, \rho^2 dx)$ with the density function $\rho(t,x)^2 := e^{2\int_0^x r(t,s)\,ds}$ (see below). We denote by $T(t)$ the closure of this operator in $L^2(\mathbb{R}, \rho^2 dx)$.

Theorem 5.3. *Let $r \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta \leq 1/2$ be a solution of the mKdV equation. Then for any $t, t' \in I$*

$$\mathtt{spec}\, T(t) = \mathtt{spec}\, T(t')$$

where $\mathtt{spec}\, T$ denotes the spectrum of an operator T. Moreover, if $\beta < 1/2$ or $r \in o_\beta(I \times \mathbb{R})$ with $\beta \leq 1/2$, then the operators $T(t)$ and $T(t')$ are unitarily equivalent.

Now, we proceed to the proof of Theorem 5.3.

Lemma 5.4. *Let $r \in C^\infty(\mathbb{R})$, $q(x) = r' + r^2$, and $\rho(x) = e^{\int_0^x r(s)\,ds}$.*

(a) The map

$$\Phi_\rho : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}), \quad u(x) \mapsto \rho(x)\,u(x)$$

extends to an isometry $L^2(\mathbb{R}, \rho^2 dx) \to L^2(\mathbb{R})$.

(b) The diagram

$$\begin{array}{ccccccc} L^2(\mathbb{R},\rho^2dx) & \supseteq & C_0^\infty(\mathbb{R}) & \xrightarrow{T_r} & C_0^\infty(\mathbb{R}) & \subseteq & L^2(\mathbb{R},\rho^2dx) \\ & & \downarrow \Phi_\rho & & \downarrow \Phi_\rho & & \\ L^2(\mathbb{R}) & \supseteq & C_0^\infty(\mathbb{R}) & \xrightarrow{L_q} & C_0^\infty(\mathbb{R}) & \subseteq & L^2(\mathbb{R}) \end{array}$$

where $T_r = -\frac{d^2}{dx^2} - 2r\,\frac{d}{dx}$ *and* $L_q = -\frac{d^2}{dx^2} + q$, *is commutative.*

Proof of Lemma 5.4. (a) is obvious. To prove (b) use the relation $\rho'/\rho = r$ to get for any $u \in C_0^\infty(\mathbb{R})$

$$T_r(u) = -(\rho^2\, u')'/\rho^2\,.$$

Hence, for any $w \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} T_r \circ \Phi_\rho^{-1}(w) &= T_r(w/\rho) \\ &= -(w''\rho - w\rho'')/\rho^2 \\ &= (-w'' + (\rho''/\rho)\, w)/\rho \\ &= \Phi_\rho^{-1} \circ L_q(w) \end{aligned}$$

where we have used that $\frac{\rho''}{\rho} = \left(\frac{\rho'}{\rho}\right)' + \left(\frac{\rho'}{\rho}\right)^2$. □

Assume that $r \in \mathcal{O}_\beta(\mathbb{R})$ with $\beta \le 1/2$. Then $q = r' + r^2 \in \mathcal{O}_\delta(\mathbb{R})$ with $\delta \le 1$. As by the Sears theorem the Schrödinger operator L_q is essentially self-adjoint the impedance operator T_r is essentially self-adjoint by Lemma 5.4. Moreover, it follows from Lemma 5.4 that the closures of both operators are unitarily equivalent. In particular,

$$\mathtt{spec}\, T_r = \mathtt{spec}\, L_q\,. \tag{70}$$

Proof of Theorem 5.3. The statement of the theorem follows from the unitary equivalence of L_q and L_r, Theorem 1.6, and the fact that the Miura map $r \mapsto r_x + r^2$ maps smooth solutions of mKdV to smooth solutions of KdV. □

6 Evolution of generalized eigenfunctions

In this section we prove that family of linear evolutions corresponding to the first order differential operators $Q_\lambda(s) = (4\lambda + 2q(s,x))\partial_x - q_x(s,x)$, where s is a real parameter between t and t' and λ is the spectral parameter, transforms any complete orthogonal system of generalized eigenfunctions of the operator $L(t)$ to such a system for the operator $L(t')$ (Theorem 6.2). Moreover, in Proposition 6.7 below, we prove that the solution of the evolution equation involving the third order differential operator $A(t) = -4\partial_x^3 + 6q(t,x)\partial_x + 3q_x(t,x)$ appearing in the classical Lax pair, can be obtained in terms of the solution of the first order evolution equation $\psi_t = Q_\lambda(t)\psi$ with the spectral parameter $\lambda \in \mathbb{R}$.

Assume that either $q \in \mathcal{O}_\beta(I \times \mathbb{R})$ with $\beta < 1$ or $q \in o_\beta(I \times \mathbb{R})$ with $\beta \le 1$ where $I = (a,b)$, $-\infty \le a < b \le +\infty$ and $0 \in (a,b)$. Assuming in addition that q is a solution of KdV consider the first order evolution equation

$$\psi_t(t) = Q_\lambda(t)\psi(t) \tag{71}$$

$$\psi|_{t=0} = \psi_0 \tag{72}$$

where $Q_\lambda(t) = (4\lambda + 2q(t,x))\partial_x - q_x(t,x)$ and $\lambda \in \mathbb{R}$ is a parameter. According to Lemma A.1 for any $\psi_0 \in C^\infty(\mathbb{R})$ the initial value problem (71)-(72) has a unique global in time solution in $C^\infty(I \times \mathbb{R})$. Moreover, if the initial data ψ_0 lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ then for any $t \in I$, $\psi(t) \in \mathcal{S}(\mathbb{R})$ and $\psi \in C^1(I, \mathcal{S}(\mathbb{R}))$ (see Lemma A.3).

Consider a Hilbert-Schmidt rigging

$$H_+ \subseteq L^2(\mathbb{R}) \subseteq H_-$$

associated to the Hilbert-Schmidt operator

$$K : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad K := (-\partial_x^2 + x^2)^{-1}$$

(see [1, Supplement 1, § 3, 4 and § 7]). One has the following chain of continuous embeddings

$$\mathcal{S}(\mathbb{R}) \subseteq H_+ \subseteq L^2(\mathbb{R}) \subseteq H_- \subseteq \mathcal{S}'(\mathbb{R}) \tag{73}$$

where $\mathcal{S}'(\mathbb{R})$ denotes the space of tempered distributions. It follows from [1, Supplement 1, § 7] that there exists a complete orthonormal system of generalized eigenfunctions

$$\{\psi(x,m) \,|\, m \in M\} \subseteq C^\infty(\mathbb{R}) \cap H_- \subseteq \mathcal{S}'(\mathbb{R}) \tag{74}$$

of $L(0)$ with generalized eigenvalues $\lambda(m)$ on a measure space M with measure μ. The complete orthogonal system of generalized eigenfunctions (74) is characterized by the following two properties (cf. [1, Supplement 1, Definition 2.4]):

(i) for any $h_+ \in H_+$ the function $m \mapsto (h_+, \varphi(m))$ on M belongs to $L^2(M,\mu)$;

(ii) the map $h_+ \mapsto F(h_+)$, $F(h_+)(m) := (h_+, \varphi(m))$, extends to a unitary operator $H \to L^2(M,\mu)$.

Here and below we use the notation $(\cdot,\cdot)$ for miscellaneous sesquilinear dualities which extend the usual L^2 inner product by continuity. So these dualities are linear with respect to the first argument and antilinear with respect to the second one. The unitary transform $F : H \to L^2(M,\mu)$ is called *generalized Fourier transform* corresponding to the system (74). Note that the functions $\psi(x,m)$ are smooth in x and satisfy the relation

$$L(0)\psi(m) = \lambda(m)\psi(m) \tag{75}$$

where $L(0)$ is the differential operator $-\partial_x^2 + q(0,x)$.

Remark 6.1. *In fact, since the multiplicity of the spectrum of $L(0)$ is at most two*[3]*, we can take $M = \mathbb{R} \sqcup \mathbb{R}$ to be a disjoint union of two copies of $\mathbb{R}$ with positive Lebesgue-Stiltjes measure on each of the copies.*

Theorem 6.2. *Denote by $\psi(t, x, m)$ the solution of the initial value problem* (71)-(72) *with the initial data $\psi_0(x) = \psi(x, m)$ and $\lambda = \lambda(m)$. Then for any $t \in I$, $\{\psi(t, x, m) \,|\, m \in M\}$ is a complete orthonormal system of generalized eigenfunctions of the operator $L(t)$ on the measure space M with the same measure μ and the same generalized eigenvalues $\lambda(m)$.*

Proof of Theorem 6.2. Consider the linear evolution equation in $\mathcal{S}'(\mathbb{R})$

$$\psi_t(t) = \hat{A}(t)\psi(t) \tag{76}$$

$$\psi|_{t=0} = \psi_0 \in \mathcal{S}'(\mathbb{R}) \tag{77}$$

where $\hat{A}(t) : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ denotes the extension by continuity of the operator (63) from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ i.e., $\forall\ \psi \in \mathcal{S}'(\mathbb{R})$ and $\forall \varphi \in \mathcal{S}(\mathbb{R})$, $(\hat{A}(t)\psi, \varphi) := (\psi, A^*(t)\varphi)$ where $A^*(t)$ is the formal adjoint to $A(t)$, and the derivative $\psi_t = d\psi/dt$ is understood in the weak topology of $\mathcal{S}'(\mathbb{R})$.

Lemma 6.3. *The initial value problem* (76)-(77) *has unique solution in $\mathcal{S}'(\mathbb{R})$.*

Proof of Lemma 6.3. Let us first prove the existence. Consider again the operator (67) extended by continuity to an isometry in $L^2(\mathbb{R})$. Then define the curve

$$\psi : I \to \mathcal{S}'(\mathbb{R}),\ t \mapsto \hat{\Psi}(t)\psi_0 \in \mathcal{S}'(\mathbb{R}) \tag{78}$$

where $(\hat{\Psi}(t)\chi, \varphi) := (\chi, \Psi(t)^*\varphi)$, $\forall\, \chi \in \mathcal{S}'(\mathbb{R})$, $\forall\, \varphi \in \mathcal{S}(\mathbb{R})$, and where $\Psi^*(t)$ denotes the adjoint operator to $\Psi(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with respect to the L^2 scalar product.[4] For any test function $\varphi \in \mathcal{S}(\mathbb{R})$ one has

$$\frac{d}{dt}(\psi(t), \varphi) = \frac{d}{dt}(\psi_0, \Psi^*(t)\varphi) = (\psi_0, \Psi^*(t)A^*(t)\varphi) = (\hat{A}(t)\psi(t), \varphi)\,.$$

Hence, $\psi(t)$ is a solution of (76)-(77).

The uniqueness follows from Holmgren's principle and the existence of a solution $\psi \in C^1(I, \mathcal{S}(\mathbb{R}))$ of the initial value problem

$$\psi_t(t) = -A(t)^*\psi(t) = A(t)\psi(t)$$

$$\psi|_{t=t_0} = \psi_0$$

for any $t_0 \in I$ ([4, Theorem 2]). □

Consider the initial value problem in $\mathcal{S}'(\mathbb{R})$

$$\psi_t(t) = \hat{Q}_\lambda(t)\psi(t) \tag{79}$$

$$\psi|_{t=0} = \psi_0 \in \mathcal{S}'(\mathbb{R}) \tag{80}$$

[3] This follows easily from (75) and the construction of generalized eigenfunctions in [1, Supplement 1,§ 5].

[4] As $\Psi(t)$ is unitary, $\Psi^*(t) = \Psi(t)^{-1}$.

where $\hat{Q}_\lambda(t)$ is the extension by continuity of the differential operator $Q_\lambda(t)$: $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ to an operator on $\mathcal{S}'(\mathbb{R})$. Arguing as in the proof of Lemma 6.3, one proves the following lemma.

Lemma 6.4. *The initial value problem* (79)-(80) *has unique solution in* $\mathcal{S}'(\mathbb{R})$.

Now, assume that $\psi(t,x,m)$ is the solution of the initial value problem (71)-(72) with initial data $\psi(x,m)$ and $\lambda = \lambda(m)$ where $\psi(x,m) \in \mathcal{S}'(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is a generalized eigenfunction of $L(0)$ with generalized eigenvalue $\lambda(m)$. Then $I \to \mathcal{S}'(\mathbb{R})$, $t \mapsto \psi(t,m)$, is the solution of (79) with initial data $\psi(m)$. Using the commutator relation (51) with $\lambda = \lambda(m)$ and arguing as in the proof of Proposition 2.2 we prove that $L_{\lambda(m)}(t)\psi(t,m) = 0$. The latter together with the relation $A(t) = Q_\lambda(t) + 4\partial_x \circ L_\lambda(t)$ with $\lambda = \lambda(m)$ implies

$$\begin{aligned} \psi(t,m)_t &= \hat{Q}_{\lambda(m)}(t)\psi(t,m) = (\hat{A}(t) + 4\hat{\partial}_x \circ \hat{L}_{\lambda(m)}(t))\psi(t,m) \\ &= \hat{A}(t)\psi(t,m)\,. \end{aligned}$$

Hence, $t \mapsto \psi(t,m)$ solves (76)-(77) with initial data $\psi_0 = \psi(m)$. By Lemma 6.3, we get that

$$\psi(t,m) = \hat{\Psi}(t)(\psi(m))\,. \tag{81}$$

As $\Psi(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary equivalence of the operators $L(0)$ and $L(t)$, one concludes from (81) and Lemma 6.5 below that $\{\psi(t,m) \,|\, m \in M\}$ is a complete orthogonal system of generalized eigenfunctions of $L(t)$ corresponding to the rigging of $L^2(\mathbb{R})$ obtained by shifting the rigging (73) via the isometry $\Psi(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. □

In the proof of Theorem 6.2 we used the following lemma. Let L_1 and L_2 be self-adjoint operators on a Hilbert space H. Let $H_+ \subseteq H \subseteq H_-$ be a rigging associated to a Hilbert-Schmidt operator $K : H \to H$ and let

$$\{\psi_1(m) \,|\, m \in M\} \subseteq H_-$$

be a complete orthogonal system of generalized eigenfunctions (associated to the rigging) with generalized eigenvalues $\lambda(m)$ on a measure set (M,μ) of the operator L_1.

Lemma 6.5. *Assume that the operators L_1 and L_2 are unitarily equivalent via the isometry $\Psi : H \to H$, and define the system $\{\psi_2(m) := \hat{\Psi}(\psi_1(m)) \,|\, m \in M\}$, where $(\hat{\Psi}\chi, \phi) := (\chi, \Psi^{-1}\phi)$, $\forall \chi \in H_-$, $\forall \phi \in \Psi(H_+)$. Then $\{\psi_2(m) \,|\, m \in M\}$ is a complete orthogonal system of generalized eigenfunctions of the operator L_2 on the same measure space (M,μ) and with the same generalized eigenvalues $\lambda(m)$.*

Proof of Lemma 6.5. The proof of the lemma is straightforward. Indeed, for any $\varphi \in H_+$ one has a.e. on M,

$$F_2(\Psi(\varphi))(m) = (\Psi(\varphi), \psi_2(m)) = (\Psi(\varphi), \hat{\Psi}(\psi_1(m))) = (\varphi, \psi_1(m)) = F_1(\varphi)(m)$$

where F_1 and F_2 denote the generalized Fourier transform corresponding to the system $\{\psi_1(m)\,|\,m\in M\}$ and $\{\psi_2(m)\,|\,m\in M\}$ respectively. In particular, one gets that for any $\phi\in\Psi(H_+)$,

$$F_2(\phi)=F_1\circ\Psi^{-1}(\phi)\,. \tag{82}$$

The later relation proves that the generalized Fourier transform $F_2:\Psi(H_+)\to L^2(M,\mu)$ can be extended by continuity from $\Psi(H_+)$ to an isometry $F_2:H\to L^2(M,\mu)$ that satisfies (82) for any $\phi\in H$. In particular, we get that

$$F_2^{-1}\circ\hat\lambda\circ F_2=\Psi\circ(F_1^{-1}\circ\hat\lambda\circ F_1)\circ\Psi^{-1}=\Psi\circ L_1\circ\Psi^{-1}=L_1$$

where $\hat\lambda$ is the multiplication operator by $\lambda(m)$ in $L^2(M,\mu)$. This completes the proof of the lemma. □

Remark 6.6. *The system of generalized eigenfunctions* $\{\hat\Psi(\psi_1(m))\,|\,m\in M\}$ *corresponds to the "shifted" rigging* $\Psi(H_+)\subseteq H\subseteq\hat\Psi(H_-)$.

Denote by $F(0)$ the generalized Fourier transform corresponding to (74) and by $F(t)$ – the generalized Fourier transform corresponding to $\{\psi(t,m)\,|\,m\in M\}$. With the notation of Theorem 6.2 we have the following corollary.

Proposition 6.7. *For any* $\psi_0\in\mathcal{S}(\mathbb{R})$ *consider the function* $\psi(t,x;\psi_0)=F(t)^{-1}\circ F(0)\psi_0$ *which can be also written as (cf. Remark 6.8 below)*

$$\psi(t,x;\psi_0):=\int\limits_M\psi(t,x,m)\,\tilde\psi_0(m)\,d\mu(m)\,, \tag{83}$$

where $\tilde\psi_0(m):=\int\limits_{\mathbb{R}}\overline{\psi(x,m)}\,\psi_0(x)\,dx$ *is the generalized Fourier transform of the initial data* ψ_0 *corresponding to the orthogonal system* (74). *Then* $\psi(t,x;\psi_0)$ *is a solution of the initial value problem* (65)-(66). *This solution is necessarily unique.*

Remark 6.8. *It follows from the construction of generalized eigenfunctions of a self-adjoint operator in [1, Supplement 1, Theorem 2.1] that there exists an isomorphism* $U(t):L^2(\mathbb{R})\to L^2(M,\mu)$ *that transforms the system of generalized eigenfunctions* $\{\psi(t,x,m)\,|\,m\in M\}$ *into a system of delta functions* $\{\chi(y,m):=\delta_m(y)\,|\,m\in M\}$ *on* M. *(The delta function* δ_m *is defined for almost every* $m\in M$, *and* $\delta_m\in(F(0)(H_+))'$ *(see below).) The function* $\tilde\psi_0$ *belongs to the space* $F(0)(H_+)$ *that corresponds to the Hilbert-Schmidt rigging* $F(0)(H_+)\subseteq L^2(M,\mu)\subseteq(F(0)(H_+))'$. *Arguing as in the proof of [1, Supplement 1, Theorem 2.1] one sees that the system* $\{\chi(y,m)\,|\,m\in M\}$ *is contained in the dual space to* $F(0)(H_+)$. *In particular, the integral in* (83) *makes sence as a continuous extension of the* L^2 *scalar product on* (M,μ). *Moreover, a trivial computation in the"model space"* $L^2(M,\mu)$ *leads to the relation* $\psi(t;\psi_0)=F(t)^{-1}\circ F(0)\psi_0$.

Proof of Proposition 6.7. It follows from (81) and (82) that $F(0) = F(t) \circ \Psi(t)$. The latter relation together with (83) give

$$\psi(t;\psi_0) := F(t)^{-1} \circ F(0)\psi_0 = F(t)^{-1} \circ (F(t) \circ \Psi(t))\psi_0 = \Psi(t)\psi_0 \,.$$

As $\Psi(t)$ is the evolution operator of (65), $\psi(t;\psi_0)$ solves (65)-(66). $\square$

Remark 6.9. *Proposition 6.7 shows that the solution of the evolution equation* (65) *involving the third order differential operator* $A(t) = -4\partial_x^3 + 6q(t,x)\partial_x + 3q_x(t,x)$*, appearing in the Lax pair, can be obtained in terms of the solution of the evolution equation* (71)-(72) *involving the first order differential operator* $Q_\lambda(t) = (4\lambda + 2q(t,x))\partial_x - q_x(t,x)$ *with parameter* $\lambda \in \mathbb{R}$*, provided we know the spectral decomposition of the operator* $L(0)$*.*

A Appendix A

In this appendix we state and prove, for the convenience of the reader, a result on the first order linear PDE, used in the main body of the paper,

$$u_t(t,x) = a(t,x)u_x(t,x) + b(t,x)u(t,x) \tag{84}$$
$$u|_{t=0} = \psi(x) \tag{85}$$

where $\psi \in C^\infty(\mathbb{R})$, $a, b \in C^\infty(\mathbb{R}\times\mathbb{R})$, and a grows for $x \to \pm\infty$ at most linearly. In addition, we prove three technical lemmas used in the proof of Theorem 1.1.

Lemma A.1. *Assume that for any* $T > 0$ *there exists a constant* $C_T > 0$ *such that for any* $|x| \geq 1$

$$|a(t,x)| \leq C_T|x| \tag{86}$$

uniformly for $t \in [-T,T]$*. Then*

(a) for any initial datum $\psi \in C^\infty(\mathbb{R})$ *there exists a unique global in time solution* $u \in C^\infty(\mathbb{R}\times\mathbb{R})$*;*

(b) if $\psi(x) > 0$ $\forall x \in \mathbb{R}$ *then* $u(t,x) > 0$ $\forall t, x \in \mathbb{R}$*.*

Proof. Clearly, the equation (84) can be rewritten in the form

$$X(u) = bu \tag{87}$$

where $X := \partial_t - a\partial_x$. Consider the ordinary differential equation

$$\dot{x} = -a(t,x), \tag{88}$$
$$x|_{t=0} = x_0. \tag{89}$$

It follows from (86) that if a solution $x(t,x_0)$ of (88)-(89) is defined on the interval $t \in (-T,T)$ for some $0 < T < \infty$ then it satisfies the a priory estimate

$$\sup_{|t|<T} |x(t,x_0)| < (1 + |x_0|)\, e^{C_T T}.$$

In particular, the latter estimate implies that for any $x_0 \in \mathbb{R}$, there exists a unique global (in time) solution $x(t, x_0)$ of (88)-(89). To prove uniqueness of a solution of (84)-(85), assume that $u = u(t, x)$ is a smooth solution. It follows from (87) that for any $x_0 \in \mathbb{R}$, the function $v(t) := u(t, x(t))$ with $x(t) := x(t, x_0)$ satisfies the differential equation $\dot{v}(t) = b(t, x(t))v(t)$, hence

$$u(t, x(t)) = \psi(x_0) e^{\int_0^t b(s, x(s))\, ds}. \tag{90}$$

As for any given $t \in \mathbb{R}$, the transformation $\mathbb{R} \to \mathbb{R}, x_0 \mapsto x(t, x_0)$, is a diffeomorphism, formula (90) defines $u(t, x)$ uniquely. At the same time, (90) defines a smooth global in time solution of (84)-(85). This proves claim (a). Claim (b) also follows from (90). $\square$

Remark A.2. *Let $a(t, x) = |x|^\alpha$ for $|x| \geq 1$, where $\alpha > 1$. Then solutions of (88) can blow up in finite time. Moreover, one can show that a solution of* (84)-(85) *is not necessarily unique. This shows that assumption (86) is essential claim* (a) *to be true.*

In the remainder of this appendix, we prove, as advertised, three technical lemmas used in the proof of Theorem 1.1 and Theorem 6.2. As above, $\mathcal{S}(\mathbb{R})$ denotes the space of functions $f : \mathbb{R} \to \mathbb{R}$ of Schwartz class.

Lemma A.3. *Assume that $q(t, x) \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ with $\delta < 1$. Then the initial value problem (49)-(50) with $\eta \in \mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$ and $s_0 \in \mathcal{S}(\mathbb{R})$ has a solution in $\mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$ which is unique in $C^\infty(\mathbb{R} \times \mathbb{R})$.*

Proof. The initial value problem (49)-(50) can be rewritten as

$$X(s) = \eta \tag{91}$$

$$s|_{t=0} = s_0 \tag{92}$$

where $X(t, x) := \partial_t - 2q(t, x)\partial_x$ and $X(s)$ denotes the derivative of s with respect to the flow of the vector field X. Denote by $\xi(t; t_0, x_0)$ the solution of the ordinary differential equation

$$\dot{x} = -2q(t, x), \tag{93}$$

$$x|_{t=t_0} = x_0. \tag{94}$$

If $t_0 = 0$ we denote the corresponding solution $\xi(t; 0, x_0)$ by $\xi(t, x_0)$. As $q \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ and $\delta < 1$ it follows that for any $0 < T < \infty$ there exists $C_T > 0$ such that for any $|x| \geq 1$ and $t \in [-T, T]$

$$|q(t, x)| \leq C_T |x|\ . \tag{95}$$

In particular, (95) implies that the solution $\xi(t; t_0, x_0)$ is defined for any $t \in \mathbb{R}$. As $q(t, x)$ is C^∞-smooth in (t, x), the solution $\xi(t; t_0, x_0)$ is unique and depends smoothly on the initial data (t_0, x_0). Moreover, for any given $t_0, t \in \mathbb{R}$, $t \geq t_0$, the transformation $x_0 \mapsto \xi(t; t_0, x_0)$, $\mathbb{R} \to \mathbb{R}$, is a diffeomorphism. Let $s(t, x)$ be a smooth solution of (91)-(92). Then the function $s(t) := s(t, \xi(t, x_0))$ satisfies the differential equation $\dot{s} = \eta(t, \xi(t, x_0))$. In particular,

$$s(t, \xi(t, x_0)) = s_0(x_0) + \int_0^t \eta(\tau, \xi(\tau, x_0))\, d\tau. \tag{96}$$

Hence, the smooth solution $s(t, x)$ of (91)-(92) is defined uniquely by the right side of (96). Equation (96) can be rewritten in the form

$$s(t, x) = s_0(\xi(0; t, x)) + \int_0^t \eta(\tau, \xi(\tau; t, x))\, d\tau\ . \tag{97}$$

Using that $s_0 \in \mathcal{S}(\mathbb{R})$, $\eta \in \mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$ together with (97) and Lemma A.4 (a) stated below one easily gets that for any $0 < T < \infty$ and for any $k \geq 0$ there exists a constant $C_{T,k} > 0$ such that for any $t \in [-T, T]$ and any x with $|x| \geq 1$

$$|s(t, x)| \leq C_{T,k} |x|^{-k}\ .$$

Differentiating equation (97) with respect to t and x we obtain that for any $k, l \geq 0$, the partial derivative $\partial_t^k \partial_k^l s(t, x)$ is a finite sum

$$\partial_t^k \partial_k^l s(t, x) = \sum_j S_j(t, x)$$

where the terms $S_j(t, x)$, with the help of Lemma A.4 below, can be shown to be of the form $S_j(t, x) = P_j(t, x) Q_j(t, x)$ with $P_j \in \mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$ and Q_j growing at most polynomially in x uniformly on compact sets of t. In particular, we get that the solution $s(t, x)$ of the initial value problem (49)-(50) lies in $\mathcal{S}_{-\infty}(\mathbb{R} \times \mathbb{R})$. The uniqueness of the solution follows from the same arguments as in in the proof of Lemma A.1. □

The following lemma is used in the proof of Lemma A.3. We use the same notation as in the proof of this lemma.

Lemma A.4. *Assume that $q(t, x) \in \mathcal{S}_\delta(\mathbb{R} \times \mathbb{R})$ with $\delta < 1$. Then the following statements hold:*

(a) For any $0 < T < \infty$ there exist constants $C_1 = C_1(T)$, $C_2 = C_2(T)$, $0 < C_1 < C_2$, and $N = N(T) > 0$ such that for any $t, t' \in [-T, T]$ and x with $|x| \geq N$

$$C_1 |x| \leq |\xi(t; t', x)| \leq C_2 |x|\ . \tag{98}$$

(b) For any $0 < T < \infty$ and for any $k, l, m \geq 0$ with $k + l \geq 1$, there exists a constant $C_{T,k,l,m} > 0$ such that for any $t, t' \in [-T, T]$ and x with $|x| \geq 1$

$$|\partial_t^k \partial_{t'}^l \partial_x^m \xi(t; t', x)| \leq C_{T,k,l,m} |x|^{\delta - m}\ . \tag{99}$$

(c) For any $0 < T < \infty$ and for any $m \geq 0$ there exists a constant $C_{T,m} > 0$ such that for any $t, t' \in [-T, T]$ and x with $|x| \geq 1$

$$|\partial_x^m \xi(t; t', x)| \leq C_{T,m} |x|^{1-m}\ . \tag{100}$$

Proof. Let $R(t,x) := -2q(t,x)$. Clearly,

$$R \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R}), \quad \delta < 1. \tag{101}$$

(a) As $R(t,x)$ satisfies for any given $0 < T < \infty$ the growth condition (95) for $x \geq 1$ and $|t| \leq T$ with some constant $C_T > 0$, the solution $\xi(t;t',x)$ (defined globally in time) of the ordinary differential equation

$$\dot{\xi} = R(t,\xi) \tag{102}$$

$$\xi|_{t=t'} = x \tag{103}$$

satisfies for any $x \geq 1$ and $t,t' \in [-T,T]$,

$$-C_T \leq \dot{\xi}/\xi \leq C_T$$

or

$$xe^{-C_T|t-t'|} \leq \xi(t;t',x) \leq xe^{C_T|t-t'|}.$$

Hence, for any $x \geq N := e^{2C_T T}$ and $t,t' \in [-T,T]$ one has $xe^{-2C_T T} \leq \xi(t;t',x) \leq xe^{2C_T T}$. Similarly one argues for $x \leq -N$ to conclude, altogether, that

$$e^{-2C_T T}|x| \leq |\xi(t;t',x)| \leq e^{2C_T T}|x|$$

for any $t,t' \in [-T,T]$ and any $|x| \geq N$.

(b) First define a class of continuous functions $\mathcal{B}^\delta \equiv \mathcal{B}^\delta(\mathbb{R}^3)$. By definition, a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ is an element in $\mathcal{B}^\delta(\mathbb{R}^3)$ iff for any $0 < T < \infty$ there exists a constant $C_T > 0$ such that for any $t,t' \in [-T,T]$ and $|x| \geq 1$

$$|f(t,t',x)| \leq C_T|x|^\delta.$$

We start by proving that for any $k,l,m \geq 0$ with $k+l \geq 1$ the function $\partial_t^k \partial_{t'}^l \partial_x^m \xi(t;t',x)$ belongs to $\mathcal{B}^\delta$. For this purpose it is convenient to consider instead of (102)-(103) the ordinary differential equation

$$\dot{y} = R(t+t',y) \tag{104}$$

$$y|_{t=0} = x \tag{105}$$

where we consider $t' \in \mathbb{R}$ as a parameter. Clearly,

$$y(t;t',x) = \xi(t+t';t',x) \tag{106}$$

where $\xi(t;t',x)$ is the solution of (102)-(103). Hence $\partial_t^k \partial_{t'}^l \partial_x^m \xi(t;t',x) \in \mathcal{B}^\delta$ if and only if

$$\partial_t^k \partial_{t'}^l \partial_x^m y(t;t',x) \in \mathcal{B}^\delta. \tag{107}$$

We will prove (107). As by assumption $R \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$, the equation (104) together with the lower and upper bounds in (98) imply that $y_t(t;t',x) \in \mathcal{B}^\delta$.

Differentiating (104)-(105) with respect to x we obtain that $y_x(t;t',x)$ satisfies the differential equation

$$(y_x)_t = R_x(t+t',y)y_x\ , \tag{108}$$

$$y_x|_{t=0} = 1 \tag{109}$$

hence,

$$y_x(t;t',x) = e^{\int_0^t R_x(\tau+t',y(\tau;t',x))\,d\tau} . \tag{110}$$

As $R_x \in \mathcal{S}_{\delta-1}(\mathbb{R}\times\mathbb{R})$ with $\delta - 1 < 0$ we get from claim (a) that for any $0 < T < \infty$ there exists a constant $C_T > 0$ such that $\forall t, t' \in [-T, T]$ and any $x \in \mathbb{R}$ one has that

$$|y_x(t;t',x)| \le C_T . \tag{111}$$

Analogously, differentiating (104)-(105) with respect to the variable t' one gets

$$(y_{t'})_t = R_x(t+t',y)y_{t'} + R_t(t+t',y) , \tag{112}$$

$$y_{t'}|_{t=0} = 0 . \tag{113}$$

By the method of the variation of parameters one obtains that

$$y_{t'}(t;t',x) = \Big(\int_0^t b(\tau)e^{-\int_0^\tau a(u)\,du}\,d\tau\Big)e^{\int_0^t a(u)\,du} \tag{114}$$

where $a(t) = a(t,t',x) := R_x(t+t',y)$ and $b(t) = b(t,t',x) := R_t(t+t',y)$. As $R_x \in \mathcal{S}_{\delta-1}(\mathbb{R}\times\mathbb{R})$ and $R_t \in \mathcal{S}_\delta(\mathbb{R}\times\mathbb{R})$ we get that $a \in \mathcal{B}^{\delta-1}$, $b \in \mathcal{B}^\delta$. Using (114) and $\delta - 1 < 0$ one concludes that $y_{t'} \in \mathcal{B}^\delta$. Differentiating successively (104)-(105) with respect to the variables t' and x one obtains an equation of the form

$$(\partial_{t'}^l \partial_x^m y)_t = R_x(t+t',y)(\partial_{t'}^l\partial_x^m y) + B(t,t',x)$$

where the inhomogeneous term B is an element in $\mathcal{B}^{\delta-m}$. Hence arguing as above one concludes that $\partial_{t'}^l\partial_x^m y(t;t',x) \in \mathcal{B}^{\delta-m}$.

In order to prove that

$$\partial_t^k(\partial_{t'}^l\partial_x^m y(t;t',x)) \in \mathcal{B}^{\delta-m} \tag{115}$$

for any $k \ge 0$ we use induction in k. By the considerations from above (115) holds for $k = 0$. Assume that $k \ge 1$ and $\partial_t^j\partial_{t'}^l\partial_x^m y \in \mathcal{B}^{\delta-m}$ for $0 \le j \le k-1$. Differentiating equation (104) with respect to t', x, and t we obtain

$$\begin{aligned}(\partial_t^{k-1}\partial_{t'}^l\partial_x^m y)_t &= \partial_t^{k-1}\partial_{t'}^l\partial_x^m(R(t+t',y)) \\ &= R_x(t+t',y)(\partial_t^{k-1}\partial_{t'}^l\partial_x^m y) + B(t,t',x).\end{aligned} \tag{116}$$

Using (98) once again together with the induction hypothesis one proves that the inhomogeneous term B is in $\mathcal{B}^{\delta-m}$. As $R_x \in \mathcal{B}^{\delta-1}$ with $\delta - 1 < 0$ and $\partial_t^{k-1}\partial_{t'}^l\partial_x^m y \in \mathcal{B}^{\delta-m}$ by induction hypothesis, formula (116) implies (115).

(c) Statement (c) follows from (98) (for $m = 0$), (111) (for $m = 1$) and then by differentiating (110) and using that $R_{xx} \in \mathcal{S}_{\delta-2}(\mathbb{R}\times\mathbb{R})$ (for $m \ge 2$). □

Let $I \subseteq \mathbb{R}$ be a finite or infinite open interval in $\mathbb{R}$. Arguing as in the proof of Lemma A.3 one proves the following lemma.

Lemma A.5. *Assume that $a \in \mathcal{O}_\beta(I\times\mathbb{R})$ and $b \in \mathcal{O}_{\beta-1}(I\times\mathbb{R})$ with $\beta \le 1$. Then the initial value problem (84)-(85) with $\psi \in \mathcal{S}(\mathbb{R})$ has a solution $u \in \mathcal{S}_{-\infty}(I\times\mathbb{R})$ that is unique in $C^\infty(I\times\mathbb{R})$. In particular, $u \in C^1(I,\mathcal{S}(\mathbb{R}))$.*

B Appendix B

For the convenience of the reader, we state in this appendix the results on the existence and uniqueness of solutions of the KdV equation

$$q_t - 6qq_x + q_{xxx} = 0 \tag{117}$$

$$q|_{t=0} = q_0 \tag{118}$$

proved in [2, 3, 4, 5] which we use in the main body of the paper.

Following earlier work of Menikoff [14], the authors of [3, 2] prove (among other things) the following theorem:

Theorem B.1. *For any $\beta < 1$ and for any initial data $q_0 \in \mathcal{S}_\beta(\mathbb{R})$ there exists a global (in time) solution $q \in \mathcal{S}_\beta(\mathbb{R}\times\mathbb{R})$ of the initial value problem (117)-(118).*

Completing results of Menikoff [14] the following uniqueness theorem is proved in [4], by use of a version of Holmgren's principle.

Theorem B.2. *For any $T > 0$, there is at most one solution of (117)-(118) in the classes of functions $q \in C^\infty([0,T]\times\mathbb{R})$ such that*

$$q(t,x) = o(|x|) \quad \textit{and} \quad \partial_x^k q(t,x) = O(1) \quad \forall k \geq 1$$

uniformly in $t \in [0,T]$.

In [5] the following theorems are proved:

Theorem B.3. *For any $\beta < 1$ and for any initial data $q_0 \in \mathcal{O}_\beta(\mathbb{R})$ there exists a global in time solution $q \in \mathcal{O}_\beta(\mathbb{R}\times\mathbb{R})$ of the initial value problem (117)-(118).*

Theorem B.4. *For any $\beta \leq 1$ and for any initial data $q_0 \in o_\beta(\mathbb{R})$ there exists a global in time solution $q \in o_\beta(\mathbb{R}\times\mathbb{R})$ of the initial value problem (117)-(118).*

Remark B.5. *According to Theorem B.2 the solutions in Theorem B.1, B.3, and B.4 are unique in the corresponding classes.*

Remark B.6. *It is likely that the methods developed in [2]-[5] can be used to prove Theorem 1.1, 1.2, and 1.3. However, the proofs will be much more difficult than the ones presented in this paper.*

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