The Soliton Resolution Conjecture

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Bill of Fare

1. Three PDE’s
2. KdV: Solitons in a Completely Integrable System
3. NLS: Solitons via Variational Argument
4. NLW: The Work of Kenig, Merle, and Collaborators
Nevertheless, it is widely believed (and supported by extensive numerics) that for many other dispersive equations (roughly speaking, those equations whose nonlinearity is not strong enough to cause finite time blowup, and more precisely for the subcritical equations), solutions with generic initial data should eventually resolve into a finite number of solitons, moving at different speeds, plus a radiative term which goes to zero. This (rather vaguely defined) conjecture goes by the name of the soliton resolution conjecture.

Terence Tao, in “Why Are Solitons Stable?”
*Bull. A. M. S.* 46 (1), 1-33
A Tale of Three PDE’s

The Korteweg-de Vries Equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

the focusing nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0$$

and the nonlinear wave equation

$$u_{tt} - \Delta u + |u|^{p-1}u = 0$$

all admit soliton solutions: travelling waves with exponentially or polynomially decaying envelopes.
Three PDE's

**KdV**

$$u_t + u_{xxx} + 6uu_x = 0$$

**NLS**

$$iu_t + \Delta u + |u|^{p-1}u = 0$$

**NLW**

$$u_{tt} - \Delta u + |u|^{p-1}u = 0$$

The *soliton resolution conjecture* states that any solution of the Cauchy problem with sufficiently decaying initial data resolves as $t \to \infty$ into a sum of soliton solutions and a “radiation term” which disperses for large times.
The KdV Soliton

Let \( u(x, t) = v(x - ct) \) in

\[
\begin{align*}
    u_t + u_{xxx} + 6uu_x &= 0
\end{align*}
\]

One recovers an ODE for \( v \):

\[
- cv'(\xi) + v'''(\xi) + 6v(\xi)v'(\xi) = 0
\]

which shows that

\[
- cv + v'' + 3v^2 = a
\]

for some constant \( a \). Multiplying by \( v' \) and integrating gives

\[
- \frac{c}{2}v^2 + \frac{1}{2}(v')^2 + v^3 = av + b
\]

which is a first-order nonlinear ODE. If we assume that \( v \) and its derivatives are rapidly decaying we can deduce that \( a = b = 0 \).
The KdV Soliton

Solving

\[(v')^2 = v^2(c - 2v)\]

we get

\[v(\xi) = \frac{c}{2} \text{sech}^2\left(\frac{\sqrt{c}}{2}(\xi - c)\right)\]

so that

\[u(x, t) = \frac{c}{2} \text{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - C)\right)\]
The KdV Soliton

\[ u(x, t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x - ct - C) \right) \]

Note that:

(i) the amplitude and speed of propagation are \textit{not} independent

(ii) the translation symmetry of KdV means that there is actually a one-parameter family “one-soliton” solutions
Complete Integrability

One can describe the large-time behavior of solutions for KdV very precisely because KdV is *completely integrable*.

For $q(x, t) \in C^\infty(\mathbb{R} \times \mathbb{R})$ let

$$L(t) = -\frac{d^2}{dx^2} + q(x, t)$$

$$A(t) = -4\frac{d^3}{dx^3} + 3 \left( q(x, t) \frac{d}{dx} + \frac{d}{dx} q(x, t) \right)$$

**Theorem** The function $q(x, t)$ solves KdV if and only if

$$\dot{L}(t) = [A(t), L(t)]$$
Complete Integrability

Recall

\[ L(t) = -\frac{d^2}{dx^2} + q(x, t) \]

The complete integrability of KdV implies that, for any solution of KdV, the operators \( L(t) \) are all \textit{unitarily equivalent}, so any invariant of the spectrum of \( L(t) \) is a constant of the motion.

Using complete integrability, one can construct \( N \)-soliton solutions for any finite \( N \) with explicit solution formulas.
Hirota Solutions*

We can generate solutions of KdV from

$$u(x, t) = \partial_x^2 [\ln \det M(x, t)]$$

where the $N \times N$ matrix (for $N$ solitons) is

$$M_{ij}(x, t) = \delta_{ij} + \frac{2\sqrt{k_i k_j}}{k_i + k_j} \exp(\xi_i + \xi_j)$$

and

$$\xi_i = k_i x - k_i^3 t + \xi^0_i$$

so that $k_i$ and $\xi^0_i$ determine the amplitude and phase of the solitons

* R. Hirota, Phys. Rev. Letters 27 (18), 1192-1194
Hirota Solutions

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Hirota Solutions

From Hirota’s formula we learn that:

- Solitons come in *multiparameter families*
- As in the 1-soliton case, one of the parameters is due to invariance of the equation under translations
- You can make great pictures and movies of solitons: https://www.youtube.com/watch?v=i7ORX97drdg
How Stable are Soliton Solutions?

It is natural to ask whether soliton solutions are stable under perturbations, i.e., if $u_0$ is the time-zero initial data for a soliton solution, $\varphi \in C_0^\infty(\mathbb{R})$, and $\varepsilon > 0$ is small, is the solution $u_\varepsilon(x, t)$ of the problem

\[
\begin{aligned}
\begin{cases}
    u_t + 6uu_x + u_{xxx} = 0 \\
    u(x, 0) = u_0(x) + \varepsilon \varphi(x)
\end{cases}
\end{aligned}
\]

“close” to a soliton solution?

- **Orbital Stability**: The solution $u_\varepsilon(x, t)$ stays at “distance” $O(\varepsilon)$ from the manifold of soliton solutions.
- **Asymptotic Stability**: The solution $u_\varepsilon(x, t)$ approaches a soliton solution, possibly with modified soliton parameters, as $t \to \infty$ and the parameter shifts are of order $\varepsilon$. 

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For *completely integrable systems*, one can often prove asymptotic stability of soliton solutions and obtain explicit formulas for the parameter shifts.

For more general classes of equations, one can often prove asymptotic stability and there is an enormous literature on the subject.
The Generalized KdV Equation*

The equation
\[ u_t + (u_{xx} + f(u))_x = 0 \]
for \( f(u) = u^p(1 + o(1)) \), \( p \geq 2 \) is called the gKdV equation. If
\[ Q''_c(x) + f(Q_c) = cQ_c \] (1)
then
\[ R_{c,x_0}(x, t) = Q_c(x - ct - x_0) \]
is a soliton solution. The number
\[ c_*(f) = \sup\{ c : \forall c' \in (0, c) : \exists Q_{c'} \text{ positive solution of (1) } \} \]
is strictly positive.

The gKdV Equation - Stability of Solitons

**Theorem** (Martel-Merle 2008) Suppose $0 < c < c_*(f)$. \exists \alpha_0 > 0 so if $u(t)$ is a global $H^1$ solution of KdV with

$$\forall t > 0, \inf_{r \in \mathbb{R}} \| u(t, \cdot + r) - Q_{c_0} \|_{H^1} \leq \alpha_0$$

then there is $t \mapsto \rho(t), t \mapsto c(t) \in (0, c_*(f))$ with

$$u(t) - Q_{c(t)}(\cdot - \rho(t)) \to 0 \text{ in } H^1(x > c_0 t/10) \text{ as } t \to \infty$$

Under stronger hypotheses, one can show $c(t) \to c_+ \text{ as } t \to \infty$. 
**Theorem (MM)** $\exists \alpha_0 > 0$ and $L_0 > 0$ so that if $$\inf_{r_j \in \mathbb{R}, r_j - r_{j-1} > L_0} \left\| u(0) - \sum_{j=1}^{N} Q_{c_j}(\cdot - r_j) \right\|_{H^1} < \alpha_0$$ then $$\forall t \geq 0, \inf_{r_j \in \mathbb{R}, r_j - r_{j-1} > L_0} \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - r_j) \right\|_{H^1} < A(\alpha_0 + e^{-\gamma t})$$
Theorem (MM) \( \exists \alpha_0 > 0 \) and \( L_0 > 0 \) so that if

\[
\forall t \geq 0, \quad \inf_{r_j \in \mathbb{R}, r_j - r_{j-1} > L_0} \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - r_j) \right\|_{H^1} < A(\alpha_0 + e^{-\gamma t})
\]

then \( \exists t \mapsto c_j(t) \in (0, c_*(f)), t \mapsto \rho(t) \) so that

\[
u(t) - \sum_{j=1}^{N} Q_{c_j(t)}(\cdot - \rho(t)) \to 0 \text{ in } H^1 \left( x > \frac{c_0 N}{10} t \right) \quad \text{as } t \to \infty
\]
Cubic NLS, One Dimension

The cubic nonlinear Schrödinger equation in one dimension

$$iu_t = -u_{xx} \pm |u|^2 u$$

is defocussing (no solitons) for the $+$ sign and focussing (solitons) for the $-$ sign. Like KdV it is completely integrable: the NLS generates an isospectral flow for the ZS-AKNS operator (on two-component vector-valued functions)

$$L = -i\sigma \frac{d}{dx} + i\sigma Q(x)$$

where

$$Q(x) = \begin{pmatrix} 0 & u(x) \\ \pm u(x) & 0 \end{pmatrix}, \quad \sigma = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
Cubic NLS, One Dimension

\[ L = -i\sigma \frac{d}{dx} + i\sigma Q(x), \quad Q(x) = \begin{pmatrix} 0 & u(x) \\ \pm u(x) & 0 \end{pmatrix} \]

For the + sign (defocusing) and \( q \in S(\mathbb{R}) \) say, the operator \( L \) is self-adjoint and has no \( L^2 \) eigenvalues.

For the − sign (focusing), the operator \( L \) is not self-adjoint and can have resonances corresponding to soliton solutions. There are \( q \in S(\mathbb{R}) \) which generate an infinite sequence of resonances converging to the real axis!
Cubic NLS, One Dimension

\[ iu_t = -u_{xx} \pm |u|^2 u \]

In the defocussing case, for \( u(0) \in H^{1,1}(\mathbb{R}) \), Deift and Zhou (2003) proved that

\[
u(x, t) \sim \frac{1}{\sqrt{t}} \alpha(z_0) e^{ix^2/4t - iv_0(z_0) \log(2t)} + O_\varepsilon \left( t^{-1/2 - \varepsilon} \right)\]

In the focussing case, Borghese, Jenkins, and McLaughlin (2016) showed that, for generic initial data (finitely many solitons)

\[
u(x, t) \sim \nu_{sol}(x, t) + t^{-1/2} f(x, t) + O \left( t^{-3/4} \right)\]

where \( \nu_{sol}(x, t) \) is a sum of one-soliton solutions, and \( f(x, t) \) contains radiation terms similar to those for the defocussing case.
The NLW is the nonlinear equation:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
    u_{tt} - \Delta u + |u|^p u = 0 \\
    u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)
\end{array}
\right.
\]

and has conserved energy

\[
E(u_0, u_1) = \frac{1}{2} \int |\nabla u_0|^2 + |u_1|^2 - \frac{1}{p+2} |u_0|^{p+2}
\]

The NLW is *critical* when \( p = \frac{4}{N-2} \) corresponding to the Sobolev embedding \( W^{1,2}(\mathbb{R}^N) \to L^{4/(N-2)}(\mathbb{R}^N) \).

For critical \( p \), if \( u \) is a solution, \( u_\lambda = \lambda^{-(N-2)/2} u(x/\lambda, t/\lambda) \) is another solution with the same energy.
For the *energy-critical NLW*

\[ u_{tt} - \Delta u + |u|^{4/(N-2)} u = 0 \]

it is known that:

(i) “Small data” gives global solutions that scatter

(ii) There exist stationary solutions, i.e., solutions \( Q \) of
\[ -\Delta Q + |Q|^{4/(N-2)} u = 0 \]
which transform to soliton-like solutions under Lorentz boosts

(iii) For large data, \((u, \partial_t u) \in C(I; \dot{H}^1 \times L^2)\) for maximal interval of existence \( I = (T_-, T_+) \)

For the rest of this talk, we’ll take \( N = 3 \) for definiteness and study

\[ u_{tt} - \Delta u + u^5 = 0 \]
Scattering Solutions

\[ u_{tt} - \Delta u + |u|^{4/(N-2)}u = 0 \]

Suppose that \( S(t) \) is the solution operator for the linear problem

\[ \nu_{tt} - \Delta \nu = 0 \]

**Theorem** There is a \( \delta > 0 \) so that if \( \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} < \delta \), there exists a unique global solution \( u \in C(\mathbb{R}; \dot{H}^1 \times L^2) \cap L^8_{xt} \) which scatters, i.e.,

\[ \|(u(t), \partial_t u(t)) - S(t)(u_0^\pm, u_1^\pm)\|_{\dot{H}^1 \times L^2} \to 0 \text{ as } t \to \infty \]

for asymptotic Cauchy data \((u_0^-, u_1^-)\) and \((u_0^+, u_1^+)\)
Stationary Solutions

\[ u_{tt} - \Delta u + u^5 = 0 \]

Once can construct stationary solutions \( u(x, t) = Q(x) \) of the equation

\[ -\Delta Q + Q^5 = 0 \]

which also occurs in the celebrated Yamabe problem for uniformization of metrics on a compact manifold. An explicit solution is

\[ Q(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2} \]

which is also the minimizer for the sharp constant in the Sobolev inequality

\[ \| u \|_{L^6(\mathbb{R}^3)} \leq C_3 \| \nabla u \|_{L^2(\mathbb{R}^3)}, \quad C_3 = \pi^{-1/2} 3^{-1/2} \left[ \frac{\Gamma(3)}{\Gamma(3/2)} \right]^{1/3} \]
Lorentz Invariance

\[ u_{tt} - \Delta u + u^5 = 0 \]

By using Lorentz invariance, we can use \( Q \) to construct travelling wave solutions

\[
Q_\ell(x, t) = Q \left( x + \left[ \frac{1}{\ell^2} \left( \frac{1}{\sqrt{1 - \ell^2}} - 1 \right) \ell \cdot x - \frac{t}{\sqrt{1 - \ell^2}} \right] \ell \right)
\]

These solutions do not scatter and provide the “soliton family” \( \Sigma \) for this problem.
Ground State Conjecture

The nonlinear ground state $Q$ establishes a threshold for blow-up phenomena to occur.

Kenig-Merle (Acta Math. 2008) established:

**Theorem** If $E(u_0, u_1) < E(Q, 0)$ then:

i) If $\|\nabla u_0\| < \|\nabla Q\|$, one has global existence and scattering

ii) If $\|\nabla u_0\| > \|\nabla Q\|$, then $T_+, |T_-| < \infty$ (blow-up in finite time)

iii) The case $\|\nabla u_0\| = \|\nabla Q\|$ does not occur
The Taxonomy of Large Data Solutions

\[ u_{tt} - \Delta u + u^5 = 0 \]

Any solution with initial data in \( \dot{H}^1 \times L^2 \) has a maximal interval of existence \((T_-, T_+)\). If e.g. \( T_+ \) is finite, then

\[ \|u\|_{L^8(\mathbb{R}^3 \times (0, T_+))} = +\infty \]

The blow up is called

**Type I** if \( \lim_{t \to T_+} \|u(t)\|_{\dot{H}^1 \times L^2} = +\infty \)

**Type II** if \( \sup_{0 < t < T_+} \|u(t)\|_{\dot{H}^1 \times L^2} < +\infty \)
Soliton Resolution for NLW

For NLW, one expects to have soliton resolution for type II solutions. Thus, if $u$ is a type II solution, one would want to show that, for a well-chosen sequence of times, $\{t_n\}$ with $t_n \to T_+$, one can write $u(x, t_n)$ as a sum of well-separated soliton solutions.
More precisely, there exists

- $J \in \mathbb{N} \cup \{0\}$,
- $Q_j$, $j = 1, \ldots, J$, $Q_j \in \Sigma$,
- $\ell_j \in \mathbb{R}^N$, $|\ell_j| < 1$, $1 \leq j \leq J$,

such that, if $t_n \uparrow T_+$ (which may be finite or infinite) and $\lambda_{j,n} > 0$, $x_{j,n} \in \mathbb{R}^N$, $j = 1, \ldots, J$ with

$$
\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \to_n \infty
$$

for $j \neq j'$ (orthogonality of the parameters) and a linear solution $v_L(x, t)$ (the radiation term) so that ...
\((u(t_n), \partial_t u(t_n))\)

\[= \sum_{j=1}^{J} \left( \frac{1}{\lambda_{j,n}^{(N-2)/2}} Q_j^j \left( \frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right), \frac{1}{\lambda_{j,n}^{N/2}} \partial_z Q_j^j \left( \frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right) \right) + (v_L(x, t_n), \partial_t v_L(x, t_n)) + o_n(1)\]
This version of the soliton resolution conjecture has been proven in the radial case $N = 3$ (DKM 2012, 2013), and general case $N = 3, 5$ $T_+ < \infty$, $u$ close to $W$ (DKM 2011).

In DKM 2012, the decomposition was proven for a well chosen sequence of times, while in DKM 2013 it was proved for any sequence of times.