# Spectral Geometry of Eigenvalues and Resonances: A Retrospective on the Work of Robert Brooks

*Dedicated To The Memory of Robert Brooks*
*1952–2002*

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**Outline**

1. Introduction: Inverse Spectral Geometry

2. Part I: The Sunada Construction and Counterexamples

3. Part II: Cheeger's Constant, Cheeger Finiteness, and Isospectral Sets of Metrics

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## Inverse Spectral Geometry: Compact Manifolds

- $X$ is a compact Riemannian manifold
- $\Delta_X$ is the Laplace-Beltrami operator on functions (with boundary conditions if $\partial X \neq \emptyset$)
- Spec$(X, g) = \{\lambda_j\}_{j=1}^{\infty}$ is the spectrum of $\Delta_X$

Two manifolds $X_1$ and $X_2$ are *isospectral* if Spec$(X_1) =$ Spec$(X_2)$

The *isospectral set* of $X$ is the set of all Riemannian manifolds isospectral to $X$

**Conjecture**: If $X$ has dimension two or three, the isospectral set of $X$ is finite.

**Inverse Spectral Geometry: Non-Compact Manifolds**

- $X$ is a complete, non-compact manifold with "simple geometry at infinity"

- $\Delta_X$ is the Laplacian on $X$

- $R(z) = (\Delta_X - z)^{-1}$ is meromorphic on a cut plane $\mathbf{C} \setminus [c, \infty)$ for some $c \geq 0$

- $R(z)$ admits meromorphic continuation to a covering of $\mathbf{C}$

Poles of the meromorphic continuation of $R(z)$ are called *scattering poles*

Examples:

- Exterior domains in $\mathbf{R}^n$
- Compactly supported metric perturbations of $\mathbf{R}^n$
- Geometrically finite hyperbolic manifolds

The *isopolar set* of $X$ is the set of all manifolds with the same scattering poles

**Conjecture**: In "low dimension" the isopolar set of a given manifold $X$ is finite.

## Part I: The Sunada Construction and Counterexamples

It is rather remarkable that all of the examples of isospectral surfaces, as well as all of the examples of Theorem 0.1, can be constructed from a single Sunada triple.

Robert Brooks and Orit Davidovich, "Isoscattering on Surfaces" (2002)

## Sunada's Method

Sunada (1985) gave a group-theoretic condition for producing pairs of isospectral manifolds. For a finite group $G$ and an element $g \in G$, denote by $[g]$ the $G$-conjugacy class of $g$.

**Definition** A finite group $G$ and subgroups $H_1$ and $H_2$ are a *Sunada triple* if

$$\#\left([g] \cap H_1\right) = \#\left([g] \cap H_2\right)$$

**Theorem** *Suppose* $(G, H_1, H_2)$ *are a Sunada triple. Suppose that* $X_0$ *is a compact Riemannian manifold and* $\phi : \pi_1(X_0) \to G$ *is an onto homomorphism. If* $X_1$ *and* $X_2$ *are finite covers of* $X_0$ *with respective fundamental groups* $\phi^{-1}(H_1)$ *and* $\phi^{-1}(H_2)$*, then* $X_1$ *and* $X_2$ *are isospectral.*

**Example** Let $G = PSL(3, \mathbb{Z}/2)$, and let

$$H_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}, \quad H_2 = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$$

Then $(G, H_1, H_2)$ is a Sunada triple.

- The outer automorphism $A \mapsto (A^{-1})^t$ takes elements in $H_1$ to $G$-conjugate elements in $H_2$
- There are generators $A$ and $B$ of $G$ so that $A$, $B$, and $AB$ have order 7
- There are generators $A'$ and $B'$ of $G$ so that $[A, B]$ has order 7

Using this Sunada triple, Brooks and Tse proved:

**Theorem** *(Brooks-Tse 1987) There are isospectral Riemann surfaces of genus* 4 *and* 6 *having constant curvature, and isospectral surfaces of genus* 3 *having variable curvature.*

In this case, $X_0$ is the sphere with three singular points, each singular point having order 7.

The homomorphism $\phi : \pi_1(X) \to G$ is given by:

$$\begin{array}{rcl} \phi(\gamma_1) & = & A \\ \phi(\gamma_2) & = & B \\ \phi(\gamma_3) & - & AB \end{array}$$

where $\gamma_i$ are loops around the singular points.

Remarkably, this construction can be revised to yield examples of isopolar two-dimensional surfaces.

**Theorem** *(Brooks-Perry 2001) There are manifolds $X_1$ and $X_2$ so that:*

*(1) $X_1$ and $X_2$ are isometric to $\mathbb{R}^2$ outside a compact set*

*(2) $X_1$ and $X_2$ have the same scattering phase and scattering poles*

The Sunada technique also yields congruence surfaces with the same eigenvalues and scattering phase.

Recall that a subgroup of $PSL(2,R)$ is a *congruence group* if it contains a subgroup of the form

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{k}\right\}$$

A *congruence surface* is a hyperbolic manifold $\Gamma \setminus \mathbb{H}_2$ where $\Gamma$ is a congruence subgroup of $PSL(2, R)$.

- Congruence surfaces have infinite discrete spectrum

- Congruence subgroups have a rigid algebraic structure

**Theorem** *(Brooks-Davidovich 2002) There exist isoscattering congruence surfaces $X_1$ and $X_2$.*

## Part II: Cheeger's Constant, Cheeger Finiteness, and Isospectral Sets

War is the continuation of diplomacy by other means.

*Carl von Clausewitz*

Analysis is the continuation of geometry by other means.

*Robert Brooks*

## Cheeger's Constant

- $X$ is a closed Riemannian manifold
- $S$ is a hypersurface dividing $X$ into two parts, $A$ and $B$
- Cheeger's constant for $X$ is

$$h(X) = \inf_S \left( \frac{\text{area}(S)}{\min(\text{vol}(A), \text{vol}(B))} \right)$$

**Theorem** *(Cheeger 1969) Let $\lambda_1(X)$ be the first nontrivial eigenvalue of $X$. Then*

$$\lambda_1 \geq \frac{1}{4} h(X)^2.$$

**Theorem** (Buser 1982) *Let* $X$ *be a closed be a closed Riemannian manifold with*

$$\text{Ricc}(X) \geq -c.$$

*There are constants* $c_1$ *and* $c_2$ *depending on* $c$ *with*

$$\lambda_1(X) \leq c_1 h + c_2 h^2.$$

Brooks realized that there was a broader relationship between isoperimetric constants and the spectrum which could be exploited to study isospectral sets.

## Background

**Theorem** *(Osgood, Phillips, Sarnak 1988)* *Let* $\{X_k\}$ *be a sequence of closed Riemannian surfaces with the same spectrum. The* $X_k$ *are diffeomorphic, and there a subsequence of the metrics on* $X_k$ *which converge in the* $\mathcal{C}^\infty$ *topology to a nondegenerate limiting metric.*

The proof exploits the heat invariants and the determinant of the Laplacian.

## Heat Invariants

If $\dim(X) = n$ and $\partial X$ is empty, then as $t \downarrow 0$:

$$\begin{aligned} \mathrm{Tr}\exp(-t\Delta_X) &= \textstyle\sum_{j=0}^{\infty} \exp(-t\lambda_j) \\ &\sim (4\pi t)^{-n/2} \textstyle\sum_{j=0}^{\infty} a_j t^j \end{aligned}$$

Here

$$\begin{aligned} a_0 &= \mathrm{vol}(X) \\ a_1 &= \tfrac{1}{6}\textstyle\int_X \mathrm{Scal}(X)\, dg \\ a_2 &= \tfrac{1}{360}\textstyle\int_X \Big[ 5\,|\mathrm{Scal}(X)|^2 - \\ &\qquad 2\,|\mathrm{Ricc}(X)|^2 + \\ &\qquad 2\,|\mathrm{Riem}(X)|^2 \Big]\, dg \end{aligned}$$

The heat invariants:

- Fix the diffeomorphism class of an oriented surface
- Are an infinite sequence of ‘nonlinear Sobolev norms’ of the curvature
- Are trivial for constant curvature metrics

The determinant of the Laplacian is defined by zeta-function regularization. In two dimensions:

- Its conformal variation can be explicitly calculated
- It is extremal on constant curvature metrics
- It is a proper function on the moduli space of a compact surface

These properties play a crucial role in the analysis of isospectral sets. What about three dimensions?

## First Steps

**Theorem** *(Brooks, Perry, Yang 1989) Suppose* $(X,g)$ *is a closed three-dimensional manifold and* $g$ *is conformally equivalent to a metric of constant negative scalar curvature. Then the set of conformal factors* $\varphi$ *with*

$$\mathrm{Spec}(X,g) = \mathrm{Spec}(X,e^{\varphi}g)$$

*is compact in* $\mathcal{C}^{\infty}(X)$*.*

**Theorem** *(Chang, Yang 1990) The same holds true if* $g$ *is conformally equivalent to a metric of zero or positive scalar curvature.*

Note that the case treated by Chang and Yang requires a *much* more subtle analysis. In each case, the conformal deformation equation is used to link integral bounds on curvature to bounds on the conformal factor. A key role is played by estimates for $\lambda_1$ in terms of the conformal factor.

## Leaving Conformal Classes: Cheeger Finiteness

Let $\mathcal{M}$ be the space of all Riemannian metrics on a fixed smooth manifold $X$.

**Theorem** *(Cheeger 1970, Gromov 1981) The set of all $n$-dimensional manifolds $(X,g)$ with*

- $|\nabla^j \mathrm{Riem}(X)|_{C^0(X)} \leq \Lambda$, $0 \leq j \leq k$
- $\mathrm{vol}(X) \geq v > 0$, *and*
- $\mathrm{diam}(X) \leq D$

*contains only finitely many diffeomorphism types, and is precompact in the $\mathcal{C}^{k+1,\alpha}$-topology on $\mathcal{M}$ for any $\alpha < 1$.*

Key issues:

- Bounding analytic Sobolev constants from above

- Bounding the diameter from below

**Theorem** *(Anderson 1991; Brooks, Perry, and Petersen 1992) The space of isospectral compact isospectral 3-manifolds $X$ for which the length of the shortest closed geodesic is bounded below:*

$$\ell_X \geq \ell > 0 \tag{1}$$

*is compact in the $\mathcal{C}^\infty$ topology. In particular, there are only finitely many diffeomorphism types of isospectral manifolds satisfying (1).*

Cheeger's and Buser's results suggest that isoperimetric constants can be controlled directly from the spectrum. Gallot (1988) showed how to estimate volumes of tubes in terms of isoperimetric constants and $L^p$ estimates on the curvature. Brooks realized that these estimates could be used to construct test functions for the low-lying eigenvalues of the Laplacian and so obtain spectral bounds on the isoperimetric constants.

## Sobolev Constants

- The 'analytic Sobolev constant':

$$C^S = \inf \left\{ \inf_{a \in \mathbf{R}} \left( \frac{\|df\|_{L^2}}{\|f + a\|_{2n/(n-2)}} \right) : f \in \mathcal{C}^\infty(X) \right\}$$

- The 'isoperimetric Sobolev constant':

$$C_S = \inf \left( \frac{\mathrm{area}(S)}{\inf(\mathrm{vol}(A), \mathrm{vol}(B))^{1-1/n}} \right)$$

## Compactness for 'Nearly' Constant Scalar Curvature Metrics

$g$-orthogonal decomposition of Ricci curvature:

$$\mathsf{Ricc}(X) = \mathsf{Scal}(X) \oplus \mathsf{Ricc}_t(X)$$

Here $\mathsf{Ricc}_t(X)$ is the traceless Ricci tensor.

Define the *reduced scalar curvature*

$$\mathsf{Scal}_r(X) = \mathsf{Scal}(X) - \frac{1}{\mathsf{vol}(X)} \int_X \mathsf{Scal}(X)\, dg$$

Define the *reduced Riemann tensor* $\mathsf{Riem}_r$ (if $\dim(X) = 3$) by

$$\mathsf{Riem}_r(X) = \mathsf{Scal}_r(X) \oplus \mathsf{Ricc}_t(X)$$

Note that $\mathsf{Riem}_r(X) = 0$ if $X$ has constant scalar curvature.

**Theorem** *(Brooks, Perry, Petersen 1994) For $n = 2$ and $n = 3$ there are constants $Q(n)$ and $K(n,k)$ so that if*

$$\lambda_k > Q(n) \int_X \mathsf{Scal}(X)\, dg$$

*and*

$$\lambda_k > K(k) \frac{\left(\int_X |\mathsf{Riem}_r(X)|^2 \, dg\right)^{1/2}}{\sqrt{\mathsf{vol}(X)}}$$

*then the set of all manifolds isospectral to $(X, g)$ is compact in the $\mathcal{C}^\infty$ topology on metrics.*

- The Sobolev constant is controlled by building test-functions for low-lying eigenvalues in ‘slices’ about the minimizing hypersurface $S$ for $C_S$

- One has $K(k) \sim k^2$ as $k \to \infty$ but by Weyl’s law $\lambda_k \sim k^{2/3}$

## Conclusion

Finally, I would like to take this opportunity to thank my colleagues, including those present at the conference and those who could not attend, for making spectral geometry a truly pleasant and exciting area in which to work. While it is my hope that the picture presented here will induce some to join this area of research, I think that a far greater inducement would be the opportunity to get to know, and to be a part of, the community which occupies itself with these questions.

*Robert Brooks*, "Inverse Spectral Geometry" (Review paper, 1997)

Perhaps Brooks' most enduring mathematical legacy is his influence upon his students and colleagues. He had a vision of mathematics as a cooperative, collaborative endeavor, and his generosity in sharing his ideas and in collaborating widely has had a profound impact upon an entire generation of spectral geometers.

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