MA 114 Worksheet #08 Solutions: Review for Exam 01

1. Find the following antiderivatives

(a) $\displaystyle\int x^2 \sin 2x \, dx$

Solution: Use Integration by Parts twice.

$$\int x^2 \sin 2x \, dx = -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{4}\cos 2x + C$$

(b) $\displaystyle\int xe^{2x} \, dx$

Solution: Use Integration by Parts.

$$\int xe^{2x} \, dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

(c) $\displaystyle\int \frac{dx}{x^2 + 2x + 10}$

Solution: Complete the square in the denominator. $\dfrac{1}{x^2+2x+10} = \dfrac{1}{(x+1)^2+3^2}$, so

$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+3^2} = \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C$$

(d) $\displaystyle\int \frac{x+3}{(x-6)(x-3)} \, dx$

Solution: Split this apart by Partial Fractions: $\dfrac{x+3}{(x-6)(x-3)} = \dfrac{3}{x-6} - \dfrac{2}{x-3}$, so

$$\int \frac{x+3}{(x-6)(x-3)} \, dx = \int \frac{3}{x-6} - \frac{2}{x-3} \, dx = 3\ln|x-6| - 2\ln|x-3| + C$$

(e) $\displaystyle\int \frac{3x+6}{x^2 - 10x + 24} \, dx$

Solution: The denominator factors as $(x-6)(x-4)$, so we will need to split this apart by Partial Fractions.

$$\frac{3x+6}{x^2-10x+24} = \frac{12}{x-6} - \frac{9}{x-4}$$
$$\int \frac{3x+6}{x^2-10x+24} = \int \frac{12}{x-6} - \frac{9}{x-4}\,dx$$
$$= 12\ln|x-6| - 9\ln|x-4| + C$$

(f) $\displaystyle\int \frac{3x^2+9x+8}{x^2(x+2)^2}\,dx$

Solution: Using Partial Fractions:

$$\frac{3x^2+9x+8}{x^2(x+2)^2} = \frac{1}{4x} + \frac{2}{x^2} - \frac{1}{4(x+2)} + \frac{1}{2(x+2)^2}$$
$$\int \frac{3x^2+9x+8}{x^2(x+2)^2}\,dx = -\frac{2}{x} - \frac{1}{2(x+2)} + \frac{1}{4}\ln|x| - \frac{1}{4}\ln|x+2| + C$$

(g) $\displaystyle\int \sin^5 x \cos x\,dx$

Solution: Here let $u = \sin x$, then $du = \cos x\,dx$ and this integral becomes:

$$\int \sin^5 x\cos x\,dx = \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6 x + C$$

(h) $\displaystyle\int \sin^2 x\,dx$

Solution:
$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(i) $\displaystyle\int \frac{dx}{x\sqrt{x^2+9}}$

Solution: Use the substitution $x = 3\tan u$, $dx = 3\sec^2 u\, du$ and

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2+9}} &= \int \frac{3\sec^2 u}{3\tan u 3\sec u}\, du \\ &= \frac{1}{3}\int \frac{\sec u}{\tan u}\, du = \frac{1}{3}\int \csc u\, du \\ &= -\ln|\csc x + \cot x| + C \\ &= -\ln\left|\frac{3+\sqrt{x^2+9}}{x}\right| + C\end{aligned}$$

(j) $\int \sqrt{16+4x^2}\, dx$

Solution: Use the substitution $x = 2\tan u$, $dx = 2\sec^2 u\, du$.

$$\begin{aligned}\int \sqrt{16+4x^2}\, dx = 2\int \sqrt{4+x^2}\, dx = 2\int \sqrt{4+4\tan^2 u}(2\sec^2 u\, du) &= 8\int \sec^3 u\, du \\ = 4(\sec u\tan u + \ln|\sec u + \tan u|) + C \\ = x\sqrt{4+x^2} + 4\ln\left|\frac{x+\sqrt{4+x^2}}{2}\right| + C\end{aligned}$$

(k) $\int x^3\sqrt{9-x^2}\, dx$

Solution: Let $u = 9 - x^2$, then $x^2 = 9 - u$ and $du = -2x\, dx$.

$$\begin{aligned}\int x^3\sqrt{9-x^2}\, dx = \int x^2\sqrt{9-x^2}\, xdx &= -\frac{1}{2}\int (9-u)u^{1/2}du \\ &= -\frac{1}{2}\int 9u^{1/2} - u^{3/2}\, du = -\frac{1}{2}\left(6u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \\ &= \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C\end{aligned}$$

(l) $\int_1^2 \frac{dx}{x\ln x}$

Solution: This is an improper integral since the integrand is not defined at the lower limit.

$$\int_1^2 \frac{dx}{x\ln x} = \lim_{a\to 1}\int_a^2 \frac{dx}{x\ln x}$$
$$= \lim_{a\to 1} \ln|\ln|x|||_a^2 = \lim_{a\to 1} \ln(\ln 2) - \ln(\ln a)$$

and this limit is undefined. Thus the integral diverges.

(m) $\displaystyle\int_1^\infty xe^{-2x}\,dx$

Solution: This is an improper integral.

$$\int_1^\infty xe^{-2x}\,dx = \lim_{M\to\infty}\int_1^M xe^{-2x}\,dx = \lim_{M\to\infty}\left(-\frac{1}{4}-\frac{x}{2}\right)e^{-2x}\Big|_1^M = \frac{3}{4}$$

2. An airplane's velocity is recorded at 5-minute intervals during a 1 hour period with the following results, in miles per hour:

$$\begin{array}{lllllll} 550, & 575, & 600, & 580, & 610, & 640, & 625, \\ 595, & 590, & 620, & 640, & 640, & 630 & \end{array}$$

Use Simpson's Rule to estimate the distance traveled during the hour.

Solution: We have that $\Delta t = 5min = \dfrac{1}{12}hr$. The distance traveled is the integral from 0 to 1 of the function with the values given above. Thus, by Simpson's Rule, this distance is approximately

$$\begin{aligned} \text{distance} &\approx \frac{\Delta t}{3}\left(f(0)+4f(\tfrac{1}{12})+2f(\tfrac{2}{12})+\cdots+4f(\tfrac{11}{12})+f(1)\right) \\ &= \frac{1}{36}(550+4\times 575+2\times 600+4\times 580+2\times 610+4\times 640+2\times 625+ \\ &\quad 4\times 595+2\times 590+4\times 620+2\times 640+4\times 640+630) \\ &= 608.611 \end{aligned}$$

3. Calculate M_6 and T_6 to approximate $\displaystyle\int_{-2}^1 e^{x^2}\,dx$.

Solution: $M_6 = 15.86539$ and $T_6 = 22.21614$.