Algebraic properties of cut ideals

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Special Session on Toric Ideals
AMS Eastern Section Meeting, Rutgers NJ

October 22, 2007
Outline

- Motivation: from cuts to toric ideals
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- Conjectures on algebraic properties
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- Cut ideals of cycles and trees
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- Cut ideals of cycles and trees
- Algebraic properties of certain graphs of interest
From cuts to toric ideals

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- $\text{Cut}(A | B) = \{ \{ i, j \} \in E(G) : i \in A, j \in B \text{ or } j \in A, i \in B \}$
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Definition (Sturmfels-Sullivant)

$$
\phi_G : K[q_{A|B} : A|B \text{ partition}] \rightarrow K[s_{ij}, t_{ij} : \{i,j\} \text{ edge of } G],
$$

\[ q_{A|B} \mapsto \prod_{\{i,j\} \in \text{Cut}(A|B)} s_{ij} \prod_{\{i,j\} \in E(G) \setminus \text{Cut}(A|B)} t_{ij} \]

The cut ideal $I_G$ is the kernel of the map $\phi_G$. It is a homogeneous toric ideal. The projective variety $X_G$ is defined by the cut ideal $I_G$. 
Object of interest: the projective variety $X_G$. 
From cuts to toric ideals

- Object of interest: the projective variety $X_G$.

- Its properties depend on the combinatorics of $G$. 
Decomposing into smaller graphs

If $G$ is a small clique sum of $G_1$ and $G_2$,

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Theorem (Sturmfels-Sullivant)

*Let $G$ be a clique sum of $G_1$ and $G_2$ with clique size at most 3.*
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**Theorem (Sturmfels-Sullivant)**

*Let $G$ be a clique sum of $G_1$ and $G_2$ with clique size at most 3.*

*Suppose that $F_1$ and $F_2$ are binomial generating sets for the smaller cut ideals $I_{G_1}$ and $I_{G_2}$. Then*
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**Theorem (Sturmfels-Sullivant)**

Let \( G \) be a clique sum of \( G_1 \) and \( G_2 \) with clique size at most 3. Suppose that \( F_1 \) and \( F_2 \) are binomial generating sets for the smaller cut ideals \( I_{G_1} \) and \( I_{G_2} \). Then

\[
M = \text{Lift}(F_1) \cup \text{Lift}(F_2) \cup \text{Quad}(G_1, G_2)
\]

is a generating set for the cut ideal \( I_G \). If \( F_1 \) and \( F_2 \) are Gröbner bases, then there exists a term order such that \( M \) is a Gröbner basis of \( I_G \).
Conjectures

Conjecture (Sturmfels-Sullivant)
The set of graphs whose cut ideals is generated in degree at most \( k \) is minor-closed for any \( k \).

The cut ideal \( I_G \) is generated by quadrics if and only if \( G \) is free of \( K_4 \) minors (that is, \( G \) is a simple series-parallel graph).

Conjecture (Sturmfels-Sullivant)
The semigroup algebra \( K[q]/I_G \) is normal if and only if \( K[q]/I_G \) is Cohen-Macaulay if and only if \( G \) is free of \( K_5 \) minors.

Gorenstein? No clear conjecture.
Conjectures

Maximal degree of a minimal generator:

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- Disjoint union of nice graphs is nice.
A phylogenetic ideal on a claw tree

Let $I_n$ denote the phylogenetic ideal for the general group-based model for the group $\mathbb{Z}_2$ on the claw tree $K_{1,n}$. 

- The ideal of phylogenetic invariants $I_n$ for this tree is the kernel of the following homomorphism between polynomial rings:

$$
\phi_n: \mathbb{C}[q_{g_1}, \ldots, q_{g_n}]: g_1, \ldots, g_n \in G \rightarrow \mathbb{C}[a_{(1)}g_1, \ldots, a_{(n+1)}g_1 + g_2 + \cdots + g_n].
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(The coordinate $q_{g_1}, \ldots, q_{g_n}$ corresponds to observing the element $g_1$ at the first leaf of the tree, $g_2$ at the second, ...)
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q_{g_1,\ldots,g_n} \mapsto a_{g_1}^{(1)} a_{g_2}^{(2)} \cdots a_{g_n}^{(n)} a_{g_1+g_2+\cdots+g_n}^{(n+1)}.
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Cycles and phylogenetics

Lemma (Petrović)

The phylogenetic ideal on the claw tree with $n$ leaves is isomorphic to the cut ideal of an $(n + 1)$-cycle.
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Lemma provides a set of properties for the cut ideals of cycles.
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**Corollary (Chifman-Petrović)**

The cut ideal of a $k$-cycle has a quadratic lexicographic Gröbner basis for $k \geq 4$. In addition, the Gröbner basis is squarefree.
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Corollary (Nagel-Petrović)

The cut varieties of cycles are Cohen-Macaulay.
Cycles: other properties

The cut ideals of cycles are not Gorenstein in general. $I_{C_4}$ is a special case: complete intersection.
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**Lemma**

*Let $n \geq 3$. The number of generators for the cut ideal of the $(n + 1)$-cycle is*

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\mu(I_{C_{n+1}}) = \binom{2^n + 1}{2} + \cdots + \binom{2^3 + 1}{2} - [3^n + \cdots + 3^3] - \left[ \binom{2^{n-1}}{2} + \cdots \binom{2^2}{2} \right].
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Note: \( \mu(I_{C_{n+1}}) = 3 S(n, 4) \).

\( S(n, 4) \) = the Stirling number of the second kind.
Trees I

Let $T$ be a tree with $n$ edges. Let $p, p+1 \in V(T)$. $T_p$ is defined to be obtained by adding a new edge $\{p, r\}$. $T_{p+1}$ is defined to be obtained by adding a new edge $\{p+1, r\}$.

Lemma
The toric varieties whose ideals are the cut ideals $I_T$ and $I_{T+p}$ have the same parametrization up to renaming variables.

Remark
Algebraic properties of cut ideals of trees depend only on the number of edges of the tree.
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Trees II

Theorem (Nagel-Petrović)

Let $T$ be a tree with $n$ edges, $n \geq 1$. Let $I_T \subset S_T$.

$$h_{S_T/I_T}(i) = (i + 1)^n.$$ 

$\dim(S_T/I_T) = n + 1$

$\deg(S_T/I_T) = n!$. 

Remark

The cut ideals of trees have a quadratic squarefree Gröbner basis. Thus, the varieties are Cohen-Macaulay.
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(All $T$ in same connected component of the Hilbert scheme.)
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\[ h(S_T / I_T) = \sum_{i \geq 0} (i + 1)^n t^i = \frac{1 + h_{1,n} t + \cdots + h_{r,n} t^{r_n}}{(1 - t)^{n+1}}, \]

where the entries of the \( h \) vector are the Eulerian numbers. In particular, the regularity is \( \text{reg}(S_T / I_T) = n - 1 \).
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Recurrence relation for the entries of the \( h \)-vector:

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h_{i,n} = (n - i)h_{i-1,n-1} + (i + 1)h_{i,n-1}.
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Corollary

The coordinate ring of the cut ideal of any tree is Gorenstein.
Consequences

Theorem (Nagel-Petrović)

If a graph can be built from trees and cycles using clique sums ($k \leq 2$), then:

- squarefree quadratic Gröbner basis (thus generated in degree 2),
- Cohen-Macaulay.
Note: not Gorenstein in general.
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In particular, the conjectures on quadratic generation and Cohen-Macaulayness are true for a large subclass of series-parallel graphs:

Example

*Outerplanar graphs.*
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**Outerplanar graphs.**

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**2-trees have the same properties.**
Minor-closed conjecture

Degrees of generators should not go up when deleting edges.

Lemma (Nagel-Petrović)

Maximal degree of a minimal generator is preserved under taking disjoint union. Cohen-Macaulayness is also preserved.
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- and one direction of conjecture on Cohen-Macaulayness.
Thank you!