Spectra and the Stable Homotopy Category

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Abstract: An introduction to the history and application of (topological) spectra, the stable homotopy category, and their relation.

1 Introduction

Why do we care about spectra and the SHC? A major goal of algebraical topology is to study the category of finite CW complexes. However, this category is big, scary, confusing, complicated, difficult. So maybe we can work with something easier. A big part of algebraic topology is coming up with invariants, but some are more rigid than others:

Unstable: fundamental group and higher homotopy groups, relative ho groups, ho groups with coeffs, localizations, completions of a space, etc [results in the homotopy category of spaces]

Stable: things which are "independent of dimension", or "stable under suspension" in some nice way.

Examples of such things:

1937 Freudenthal proves his suspension theorem: for finite CW X, Y, we have $[X,Y] \to [S^1 \wedge X, S^1 \wedge Y] \to \dots$ stablilizes to isomorphisms after finitely-many steps

1950s Stable homotopy groups: $\pi_n^s(X) = \lim_{N\to\infty} \pi_{n+N} S^N \wedge X$

1952 Suspension isomorphism in (co)homology: $E^n(X) \cong E^{n+1}(S^1 \wedge X)$

Goal: produce a "stable homotopy category" whose objects were some stablized analog of spaces, in which these results live.

2 Intuition from Rings

Given a commutative ring R, we can look the category of chain complexes. We have a notion of homotopy of chain maps, and we can construct the resulting homotopy category. But this really isn't the category algebraists want to study; e.g. if we think of two R-modules M and N as boring chain complexes concentrated in degree 0, the homology of their tensor product should be $\mathrm{Tor}_*^R(M,N)$. Formally, we only care about the homology groups of the chain complexes, so we call a chain map a *quasi-isomorphism* if it induces an isomorphism on homology groups, and the *derived category* is obtained by adjoining formal inverses to the quasi-isomorphisms. We can do this nicely by introducing the idea of a "cell" R-module, somewhere where quasi-iso IFF homotopy equivalence (think Whitehead Theorem), and every chain complexe is quasi-iso to a cell R-module. Then the derived category will be equivalent ot the homotopy category of cell R-modules.

We will have a sphere spectrum S as our "ring" R; spectra will be (weak) analogues of chain complexes; we'll have a boring definition of homotopy equivalence, and a finer notion of weak equivalence. There will be CW spectra, and a Whitehead Theorem, and everything.

3 First Definitions of the SHC

Try #1: Spanier-Whitehead category (1953).
Objects are finite CW, morphisms $\{X, Y\} = \mathrm{colim}[S^n \wedge X, S^n \wedge Y]$.

1. Additive
2. Symmetric monoidal under a smash product

Problem: not triangulated/ suspension isn't an isomorphism/ we don't have Spanier-Whitehead/Alexander Dualty. So this isn't quite right: we want the suspensions really to work nicely.

Try #2: Objects are $\Sigma^{\infty+n}A$, A finite CW; morphisms $[\Sigma^{\infty+n}A, \Sigma^{\infty+m}B] = \lim_{N\to\infty}[\Sigma^{N+n}A, \Sigma^{N+m}B]$. Has the following properties:

1. Triangulated (in particular, suspension is a self-equivalence)
2. Additive
3. Symmetric monoidal under a smash product
4. "First approximation to homotopy theory"

Really, this is the category we're want - finite CW complexes up to suspension. But working in this category is *hard*, as it is not quite a good category. So maybe we can place this category inside other larger and better categories, where we can study it more effectly. In particular, we actually want to use (co)homology theories. But those are represented by infinite loop spaces, which are hideously huge. So in order for this theory to be "complete" in some sense, we need to include those cohomology objects as well.
Solution: In his 1964 Ph.D. thesis, Boardman introduced the first real usable version of the SHC by (formally) adding homotopically well-behaved colimits into the SW-category. Unfortunately, he never published his work, and moreover his technology isn't something we currently use. Fortunately, George Whitehead (as opposed to JHC Whitehead above) and Spanier developed new technology to deal with cohomology theories, and Adams (and May?) used it to form the current model for Boardman's category. Moreover, in 2001, Schwede and Shipley proved this category was essentially unique (so Boardman picked the right thing!)

4 Generalized (co)homology Theories and Spectra

To motivate this, let's look at a couple of examples of the work that went on in the following decade that led to a more effective SHC:

1954 Thom: classification of smooth n-manifolds up to cobordism. The set N_n of cobordism classes is isomorphic to $\pi_n^s TO$ where $TO(q)$ is the Thom space of the universal q-plane bundle $\xi_q : EO(q) \to$

$BO(q)$. Moreover, Whitney sum corresponds to maps $BO(q) \times BO(r) \to BO(q+r)$ correspond to $TO(q) \wedge TO(r) \to TO(q+r)$, which induce pairings of homotopy groups

$$\pi_{m+q}(TO(q)) \otimes \pi_{n+r}(TO(r)) \to \pi_{m+n+q+r}(TO(q+r))$$

which correspond to the cartesian product of manifolds

1960 Adams: the only possible dimensions of a normed linear algebra over $\mathbb{R}$ are 1,2,4,8 (which was known), as a small consequence of solving the Hopf invariant 1 problem, by translating the issue into a problem in mod 2 cohomology involving the Steenrod operations, which are stable. [aside: cup products, which are very instable - cup of two classes in $\tilde{H}^*(S^1 \wedge X)$ is 0]

1961 Atiyah and Hirzebruch invent topological K-theory, "extending" Bott periodicity: $\pi_n(BO) \cong \pi_{n+8}(BO)$. $KO(X)$ = Grothendieck group of the semi-group, under the Whitney sum, of the set of bundles over X of all dimensions. There exists a reduced version, and we see $\tilde{KO}(X) \cong [X, BO \times \mathbb{Z}]$. We can turn this into a cohomology theory via Bott, satisfying $\tilde{KO}^n(X) \cong \tilde{KO}^{n+8}(X)$. Tensor product of bundles gives rise to a multiplication in KO-theory (analogus to the cup product).

Let's explore this last idea more; in particular, let's build up to Brown Representability. Eilenberg and MacLane had constructed their so-named spaces $K(\pi, n)$ for abelian groups π, where $\pi_q(K(\pi, n)) = \pi$ if $q = n$ and 0 otherwise. Then we have (late 1950s) $\tilde{H}^n(X; \pi) \cong [X, K(\pi, n)]$. Moreover, the suspension isomorphism for reduced cohomology theories corresponds to the fact that $K(\pi, n) \simeq \Omega K(\pi, n+1)$. Moremoreover, if π is a commutative ring R, the cup product on $\tilde{H}^*(X; R)$ is induced by the diagonal map on X and pairings $K(R, m) \wedge K(R, n) \to K(R, m+n)$.
Example 2: $MO(n) = \mathrm{colim}(TO(n) \to \Omega TO(n+1) \to \Omega^2 TO(n+2) \to \ldots)$, where this map is adjoint to the pullback over $i_q : BO(q) \to BO(q+1)$. We can define a cohomology theory $\tilde{MO}^n(X) = [X, MO(n)]$ or $[X, \Omega^{-n} MO(0)] = [\S^n \wedge X, MO(0)]$. The Whitney sum pairing of Thom spaces induces a product on the cobordism theory $\tilde{MO}^*(X)$.
Example 3: Define $KO(8j - i) = \Omega^i(BO \times \mathbb{Z})$; Bott periodicity gives homotopy equivs $KO(n) \simeq \Omega KO(n+1)$, and define $\tilde{KO}^n(X) = [X, KO(n)]$; the tensor product pairings yields products on $\tilde{KO}^*(X)$.

Theorem 1 (Brown Representability, 1962. Corollary)**.** *For an additive cohomology theory E, there is a sequence of pointed CW-complexes K_n with homotopy equivalences $\tilde{\sigma}_n : K_n \to \Omega K_{n+1}$ and classes $u_n \in E^n(K_n)$ such that*

$$[X, K_n]_* \to \tilde{E}^n(X) \qquad [f] \mapsto f^*(u_n)$$

is a bijection for all pointed CW-complexes X, and the maps $\tilde{\sigma}_n$ are compatible with the suspension isomorphisms.

This motivates (George) Whitehead's definition of spectra (and, from there, the SHC) (though I will use Adam's terminology, since this is what is currently used):

Definition 2. A *prespectra* is a sequence of pointed (spaces with the homotopy type of) CW complexes E_n with structure maps $\sigma_n : E_n \wedge S^1 \to E_{n+1}$, and morphisms are sequences of maps compatible with the structure map up to homotopy. An Ω *spectrum* is a prespectrum where the adjoint structure maps $\tilde{\sigma}_n : E_n \to \Omega E_{n+1}$ are weak equivalences. A (genuine) *spectra* is a prespectra such that the $\tilde{\sigma}_n$ are homeomorphisms.

Examples: All the above guys; suspension spectra $\Sigma^\infty X$ with $(\Sigma^\infty X)_n = S^n \wedge X$, etc.

Properties:

1. We have a forgetful functor U from spectra to prespectra, and a left adjoint L.
2. We can smash a prespectrum E with a based space X by $(E \wedge X)_n = E_n \wedge X$; and a spectrum with a based space by $(E \wedge X) = L(UE \wedge X)$. This allows us to define *homotopies of maps of spectra*
3. We have an evaluation functor $Ev_n : E \mapsto E_n$; and a left adjoint "shift desuspensions functor" or "free functor" F_n.

Definition 3. We define sphere spectrums $S^n = \Sigma^\infty S^n$ and $S^{-n} = F_n S^0$.
We define the homotopy groups of spectra by $\pi_n(E) = [S^n, E]$ (equivalently, $= \lim_{N\to\infty}[S^{n+N}, E_n]$).
A map of spectra is called a *weak equivalence* if it induces an iso of homotopy groups. Define the *homotopy category of spectra*, which identifies homotopic maps. Define the *stable homotopy category* to be the category obtained from the homotopy category of spectra by adjoining formal inverses to the weak equivalences.

Properties:

1. Adjoint Functors $\Sigma^\infty : T \to SHC : \Omega^\infty$
2. SHC is trianglated (in particular, suspension is an equivalence of categories)
3. SHC is additive (in particular, morphism sets in SHC are (graded) abelian groups, and composition is (graded and) bilinear)
4. Contains arbitrary products and coproducts
5. Closed symmetric monoidal (✩)
6. The subcategory of finite CW-spectra is isomorphic to SW-cat
7. Objects represent all cohomology theories (essentially uniquely)
8. Every spectrum is weakly equivalent to a CW spectrum, and SHC is equiv to the homotopy category of CW spectra.

This is Boardman's category. This is the full glory of the SHC.
Remark: The linear part of the identity (in Goodwille calculus) is $\Omega^\infty\Sigma^\infty$, which is weakly equivalent to $\lim_{N\to\infty}\Omega^N\Sigma^N$.

(Intermediary Cateogories: Whitehead's category of generalized cohomology theories, Puppe and Adam's had ones where the structure maps were inclusions of complexes, etc)

✩Boardman's Ph.D. thesis in 1964 really gave the first complete and usable description of the SHC, and it is *the* SHC all other models get compared to. Defining the symmetric monoidal structure is very non-trivial: there is no obvious choice for $E \wedge E'$, and naive attempts via pairings $E_n \wedge E_m \to E_{n+m}$ run into problems of " permutations of suspensions coordinates". Boardman instead constructs an external smash product, and then internalizes it via some interesting construction (people still use this idea, and mainly internalize via a left Kan extension). However, his construction only yields a nice homotopy category: the smash product is commutative, associative, and unital only up to homotopy. Which means, while we have these nice point-set level spectra objects to work with, a lot of the constructions we want to work with (namely the smash product) don't play nice until we go up to the homotopy category.

This makes life hard, and led to many years of sadness, culminating with Lewis's paper in 1991 which basically said "there is no perfect category of spectra". May and others didn't think they would ever find a category of spectra which had the above properties of the SHC but on the lower point-set level.

5 Modern Categories of Spectra

Life is good! We have a very large number of categories of spectra which have closed symmetric monoidal structures, and many of the other structures and properties of the SHC, such that their homotopy category is the SHC and that construction lets the SHC inherit the structures from the point-set models. - Model categories - Symmetric Spectra

References

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