MA162: Finite mathematics

Paul Koester

University of Kentucky

December 2, 2013

Schedule:

- Web Assign assignment (Chapter 5.1) due on Friday, December 6 by 6:00 pm.
- Web Assign assignment (Chapter 5.2) due on Tuesday, December 10 by 6:00 pm.
- Web Assign assignment (Chapter 5.3) due on Friday, December 13 by 6:00 pm.
- Exam 4 on Monday, December 16, 8:30 pm to 10:30 pm.

Today is Chapter 5.1: Compound Interest

Time Value of Money

- It is generally accepted that money has a time value
- As a consequence, if you had to choose between accepting a fixed sum of money either today or one year from now, you would (most likely) choose to accept the money today.
- To coerce you to accept the money at a later date, you would generally need to be offered additional money
- The additional money is called interest

Why does money have a time value?

- Inflation?

- Inflation is concerned with a change in general price levels. In one year, $1000 will generlly buy *less* than the same $1000 today. Perhaps I will need $1013 on year from now to buy the same goods that I could purchase today for $1000.

- However, *inflation* is not responsible for *time value of money.* Interest is generally expected to be paid even in non-inflationary economies.

Why does money have a time value?

- Risk?

- If I lend you \$1000 for a year, I take on the risk that you might not pay me back, so it makes sense for me to charge you a small fee since I am taking on the risk that you default.

- However, *interest* is NOT the same as a *risk premium*. Interest would be expected to be paid even if there was no risk of default.

Time value of money as a measure of usefulness

- Neither inflation nor risk are (solely) responsible for the time value of money.
- Money has a time value simply because having money today is more useful than having the same amount of money one year from now.
- Economists generally use the more techinical term “utility” as opposed to “usefulness” but they basically mean the same thing.

Computing Interest

- Several different methods for computing interest are used in practice
- We will restrict attention to compound interest
- Under compound interest, interest earned in previous periods may earn interest.

Compound Interest

- Let P denote the initial investment, also called the principal
- Let t denote the Term, or number of years
- Let r denote the nominal interest rate per year
- Let m denote the number of conversion periods per year
- Let i denote the period interest rate
- Let n denote the number of conversion periods in the term
- Let A denote the accumulated value at the end of n periods

Compound Interest, future value form

Then

$$i = \frac{r}{m},$$
$$n = mt,$$

and

$$A = P(1+i)^n,$$

sometimes written as

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

This is to be used if you know what you start with and want to find what you end with.

Compound Interest, present value form

Sometimes you will know what you need to end with and need to determine what you should start with.
Solving for P gives

$$P = A(1+i)^{-n} = \frac{A}{(1+i)^n}$$

sometimes written as

$$P = A\left(1+\frac{r}{m}\right)^{-mt} = \frac{A}{\left(1+\frac{r}{m}\right)^{mt}}$$

Compound Interest Example 1

Find the accumulated amount A if the principal $P = \$3,200$ is invested at the interest rate of $r = 1.2\%$ for $t = 3$ years, given that

- interest is compounded annually
- interest is compounded quarterly
- interest is compounded daily

Compound Interest Example 2

Juan wants to have \$3000 saved up six months from now to make a down payment on a new car. Suppose that he can invest money today at nominal interest rate $r = 0.9\%$. How much should he invest today, given that

- interest is compounded semi-annually

- interest is compounded monthly

Comparing interest rates

An interest rate depends on both the nominal interest rate and the frequency of compounding.

Suppose you can lend $\$10,000$ for two years. One borrower offers to pay you 6% compounded semiannually and a second borrower offers to pay you 5.95% compounded monthly. Who should you lend your money to?

Effective rate of interest

The effective rate of interest, or annual yield, provides us with a simple way to compare two nominal interest rates.

$$r_{eff} = \left(1 + \frac{r}{m}\right)^m - 1$$

Compute r_{eff} for the two nominal interest rates in the previous example.

Non-constant Interest Rates and Time Diagrams

Randolf borrowed $\$10,000$ from a bank 4 years ago. It is now time to repay the loan. His loan was an adjustable rate loan (ARM), meaning the bank could change the interest rate at certain points. Suppose his nominal interest rate was $r = 2\%$ compounded semi-annually for the first 2 and a half years. Suppose the interest rate was increased to $r = 4\%$ compounded semi-annually between year 2.5 and the beginning of year 3. Suppose the interest rate dropped to $r = 3\%$ compounded annually for the last year. How much does Randolf owe?

Formulas for Non-constant interest rates?

Your best strategy for dealing with non-constant interest rates is to draw a time diagram like we did above, and identify time segments on which the interest rate is constant.

Multiple payments

Samantha opened a savings account 5 years ago, and invested $\$1,200$ when she opened it. After two years, she withdrew $\$400$. After another two years, she deposited another $\$750$. Suppose her account earned 3% nominal interest compounded annually for the first three years, and she earned 1.2% nominal interest compounded monthly during the last two years. What is the value of her account today?

More on Varying Interest Rates (Optional)

In more advanced treatments, one may have to consider the force of interest $\delta(t)$. The accumulated value from time $s = 0$ to $s = t$ is then

$$A = Pe^{\int_0^t \delta(s)ds}$$

In the case of compound interest, $\delta(s)$ is just a constant, called the continuously compounded rate of interest, ρ. In this case, $\int_0^t \rho ds = \rho \cdot t$