MA162: Finite mathematics

Paul Koester

University of Kentucky

Solutions.

December 4, 2013

SCHEDULE:

- Web Assign assignment (Chapter 5.1) due on Friday, December 6 by 6:00 pm.
- Web Assign assignment (Chapter 5.2) due on Tuesday, December 10 by 6:00 pm.
- Web Assign assignment (Chapter 5.3) due on Friday, December 13 by 6:00 pm.
- Exam 4 on Monday, December 16, 8:30 pm to 10:30 pm.

Today is Chapter 5.2: Annuities

Annuities

- The previous section introduced us to the time value of money and compound interest.
- The last few examples dealt with several payments, spread out over time.
- In this section we will find a formula for dealing with several payments spread out over time, provided the payments and time intervals are regularly spaced.

Saving for a trip

- You put \$100 into a savings account at the end of each month for the next 2 years.
- Your savings account pays 1.2% nominal interest compounded monthly.
- How much will you have at the end of the two years?

Saving for a trip: some estimates

Before we find the exact value, lets look at a couple estimates.

- If you earned NO interest, then you would accumulate \$2400 (\$100 each month for 24 months) Interest will HELP! Your accumulated value should be greater than the \$2400
- If you invested the entire \$2400 today, you would have

$$2400\left(1+\frac{0.012}{12}\right)^{24}=2458.27$$

  Some of your money will be invested for less than the full 24 months, so your accumulated value can't be greater than \$2458.27
- Half of the payments will earn interest for at least 12 months, half of the payments will earn interest for fewer than 12 months. So, if we assume the entire \$2400 earns interest for exactly 12 months should give us a reasonable approximation,

$$2400\left(1+\frac{0.012}{12}\right)^{12}=2428.96$$

Saving for a trip: exact values

Your first payment is made at the end of the first month, so it will earn interest for 23 months: $100(1.001)^{23}$

Your second payment is made at the end of the second month, so it will earn interest for 22 months: $100(1.001)^{22}$

see time diagram on next page

Your last payment is made at the end of the 24th month, so it will earn interest for 0 months: $100(1.001)^0 = 100$

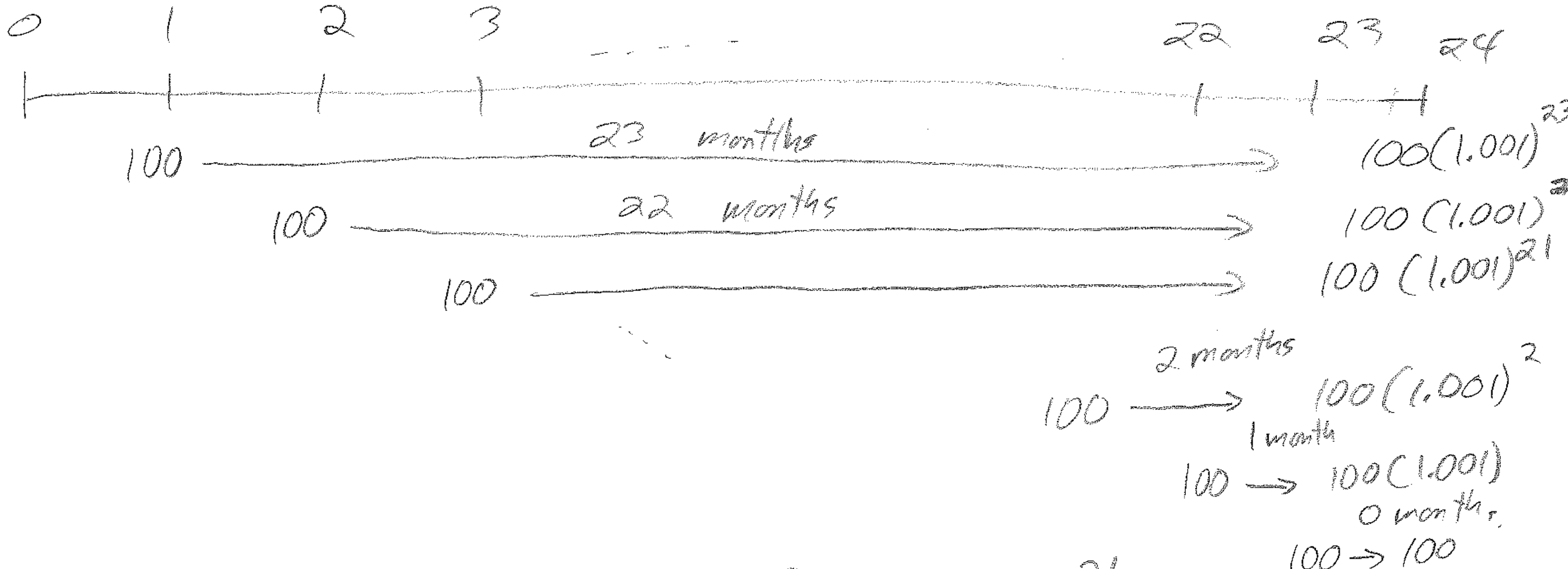
The total is then

$$\sum_{k=1}^{24} 100 \left(1 + \frac{0.012}{12}\right)^{24-k}$$

This can be computed with persistence (compute 24 things then add them...)

Fortunately, there is a formula:

$$100 \frac{\left(1 + \frac{0.012}{12}\right)^{2 \cdot 12} - 1}{\frac{0.012}{12}} = 2427.80$$



$$FV = \underbrace{100(1.001)^{23} + 100(1.001)^{22} + 100(1.001)^{21} + \cdots + 100(1.001)^{2} + 100(1.001)^{1} + 100(1.001)^{0}}$$

This could be computed quickly with a spreadsheet program.

OR, with the formula

$$100 \cdot s_{\overline{24|}\,0.001} = 100\,\frac{(1.001)^{24}-1}{0.001} = 2427.80$$

Annuity

An simple ordinary annuity certain consists of a sequence of payments so that

- the payments are all equal, say R
- each payment is made at the end of each investment period (ordinary)
- the investment periods are of equal length (for example, once per month, or once per year, etc)
- the interest conversion period is equal to the length of the investment periods (simple)
- the payments persist for a fixed number of terms (certain)

We only consider annuity certains in this course, so we will usually drop the word certain

Annuity

- We only consider annuity certains in this course, so we will usually drop the word certain
- The alternative to an annuity certain is a contingent annuity, in which payments are made provided certain conditions are met. A retirement pension is an example of a contingent annuity, in which payments are made to a pensioneer, provided the pensioneer is still alive. Valuing contingent annuities requires combining ideas from time value of money with mortality probabilities.
- We only consider simple annuities in this course.
- The alternative to an simple annuity is a complex annuity: these are annuities in which the interest conversion period and the length of the investment periods do not match. Complex annuities can be converted to to simple annuities by replacing the given interest rate with an appropriate effective interest rate.

Annuity

- We only consider ordinary annuities in this course.
- The alternative to an ordinary annuity is an annuity due, in which the payments are made at the beginning of the periods.

Annuity Formulas

The future value of a simple ordinary annuity with n level payments of R dollars each period, paid at the end of each period into an account that earns interest at the rate of i per period is

$$S = R\frac{(1+i)^n - 1}{i}$$

The present value of this annuity is

$$P = R\frac{1-(1+i)^{-n}}{i}$$

See the appendices to Chapters 5.1 and 5.2 of the text for instructions on how to use a TI-83 or TI-84 calculator or MS Excel to help compute with these formulas.

On exams, you may leave answers in "unsimplified form", for example, you could leave answer as

"$P = 450 \cdot \frac{1-(1+0.003)^{-60}}{0.003}$"

Actuarial Annuity Symbols

Many formulas in finance involve expressions like $\frac{(1+i)^n-1}{i}$, so it is conventient to have short-hand notations for these.

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

and

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

In this new notation,

$$S = Rs_{\overline{n}|i}$$

and

$$P \cancel{A} = Ra_{\overline{n}|i}$$

On exams, you can leave answers with unsimplified annuity symbols, like $S = 200 s_{\overline{24}|0.01}$

Annuity Calculation

$n = 40$
40 years
$i = 0.12$

John is 25 years old and wants to start saving for retirement at age 65. He decides to put at least \$3000 into an individual retirement account (IRA) at the end of each year for the next 40 years. Due to IRS laws, he cannot invest more than \$5000 into the account in any given year. Suppose the account earns 12% interest compounded annually. (12% is the average rate of return on large cap stocks over the last 80 years)

Future Value

- What is the least amount he will have saved at the end of the 40 years?

$R = 3000$, so future value: $S = \boxed{3000\, s_{\overline{40}|0.12}}$

$$= 3000\,\frac{1.12^{40}-1}{0.12} = 2,301,274.26$$

- What is the most he will have saved at the end of the 40 years?

Still Future value after 40 years, but now make largest payment each period

$R = 5000$

$$\boxed{S = 5000\, s_{\overline{40}|0.12}}$$

$$= 5000\,\frac{1.12^{40}-1}{.12} = 3,835,457.10$$

On exam, you can leave answers in these forms.

Annuity Calculation

John decides he is too young to start saving, and decides that he will start saving for retirement when he turns 31.

- What is the least amount he will have saved at the end of the 34 years?

$$\boxed{3000\, s_{\overline{34}|0.12}} = 3000 \cdot \frac{(1.12)^{34}-1}{.12}$$

$$= 1,153,562.94$$

- What is the most he will have saved at the end of the 34 years?

$$\boxed{5000\, s_{\overline{34}|0.12}} = 5000 \cdot \frac{1.12^{34}-1}{.12}$$

$$= 1,922,604.90$$

Notice that, by delaying retirement by 6 years, John's retirement savings was cut in half!

Financing a Home

Present Value

Mia has accumulated \$25,000 in a savings account that she intends to use as a down payment toward the purchase of a new house. She can get a 20 year loan for 7.2% nominal interest compounded monthly. After reviewing her finances, she determines that she can afford a monthly payment of \$1200 per month. What is the most expensive house she can afford to buy, assuming the monthly payments are made at the end of each month?

$R = 1200$, $t = 20$, $m = 12$, $n = 12 \cdot 20 = 240$

$r = 0.072$, $i = \frac{0.072}{12} = 0.006$

Mia's loan: $P = R\, a_{\overline{n}|i} = 1200\, a_{\overline{240}|0.006}$

$= 1200 \, \frac{1 - (1.006)^{-240}}{0.006} = \underbrace{152{,}410.12}_{\text{How much can Mia afford to borrow}}$

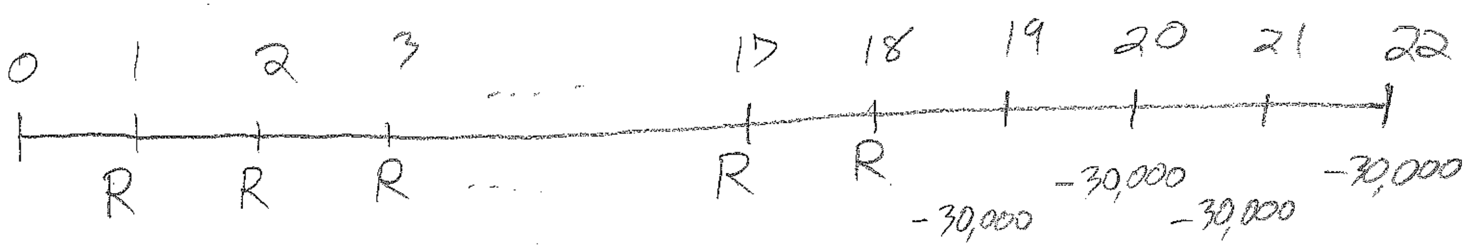
But she has extra 25,000. So most expensive home she can buy is worth

$$\boxed{25{,}000 + 1200\, a_{\overline{240}|0.006}} = 177{,}410.12.$$

On Exam, can leave in this form

Saving for College

The Brown's have a new born daughter and have decided to start saving for her college education. They estimate that she will attend school for 4 years and they estimate her schooling will cost \$30,000 per year. They will deposit a fixed amount of money into a savings account at the end of each year for 18 years. The \$30,000 will come due at the end of each of the 19th, 20th, 21st, and 22nd years. They estimate that they will be able to invest at a nominal interest rate of 4% compounded annually during the entire 22 years. How much money will they need to deposit each year?



Set up an equation of Value at time $t=18$.

Accumulated Value of 18 payments is

$R \cdot s_{\overline{18}|0.04}$

Present Value (at $t=18$) of the 4 payments of 30,000 is

$30{,}000\ a_{\overline{4}|0.04}$

So $R\ s_{\overline{18}|0.04} = 30{,}000\ a_{\overline{4}|0.04}$

OK to leave like this on exam.

So $\boxed{R = \dfrac{30{,}000\ a_{\overline{4}|0.04}}{s_{\overline{18}|0.04}}}$

$$R\ \frac{1.04^{18}-1}{.04} = 30{,}000\ \frac{1-1.04^{-4}}{.04}$$

$$R \cdot 25.6454 = 108896.8567$$

So $R = \dfrac{108896.8567}{25.6454} = 4246.25$