MA162: Finite mathematics

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Schedule:

- Sixth Web Assign assignment (Chapter 2.4) due on Tuesday, September 24 by 6:00 pm.
- Seventh Web Assign assignment (Chapter 2.5, 2.6) due on Friday, Septemeber 27 by 6:00 pm.
- Exam 1 on Monday, September 30 from 5:00 pm to 7:00 pm

Today we cover Chapter 2.5: Matrix Multiplication. How we multiply matrices is not nearly as natural as how we add matrices

How NOT to Multiply Two Matrices

- We add matrices by adding corresponding entries together.

- Why not do the same for multiplication of matrices?

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 & 2 \cdot 2 & 3 \cdot 2 \\ 4 \cdot 1 & 3 \cdot 5 & 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 6 \\ 4 & 15 & 6 \end{bmatrix}?$$

- This is a logically sound idea, and the computation is not too difficult.

- Unfortunately, this sort of "matrix multiplication" isn't really useful in practical applications.

Correct Way to Multiply Two Matrices

- Suppose

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

- Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 2 & 1 \cdot 3 + 2 \cdot 1 & 1 \cdot 2 + 2 \cdot 0 & 1 \cdot 1 + 2 \cdot (-1) \\ 3 \cdot 4 + 4 \cdot 2 & 3 \cdot 3 + 4 \cdot 1 & 3 \cdot 2 + 4 \cdot 0 & 3 \cdot 1 + 4 \cdot (-1) \\ 5 \cdot 4 + 6 \cdot 2 & 5 \cdot 3 + 6 \cdot 1 & 5 \cdot 2 + 6 \cdot 0 & 5 \cdot 1 + 6 \cdot (-1) \end{bmatrix}$$

$$\begin{bmatrix} 8 & 5 & 2 & -1 \\ 20 & 13 & 6 & -1 \\ 32 & 21 & 10 & -1 \end{bmatrix}$$

Facts About Matrix Multiplication

- The "inner dimensions" must agree. In order for AB to be defined,

$$\# \text{ columns of } A = \# \text{ rows of } B$$

- If AB is defined, then

$$\# \text{ rows of } AB = \# \text{ rows of } A$$

- If AB is defined, then

$$\# \text{ columns of } AB = \# \text{ columns of } B$$

Matrix Multiplication: A motivating example

- A farmer has 100 acres of land. She grows corn on 50 acres, wheat on 30 acres, and soybeans on 20 acres.
- Her costs per acre are: \$700 per acre of corn; \$765 per acre of wheat; \$500 per acre of soybeans.
- Revenue generated per acre: \$651 per acre of corn; \$523 per acre of wheat; \$543 per acre of soybeans.
- What is her total cost?

$$\text{Cost} = (\text{Cost per acre corn})\cdot(\text{acres of corn}) + (\text{Cost per acre wheat})\cdot(\text{acres of wheat}) + (\text{Cost per acre soy})\cdot(\text{acres of soy})$$

$$= 700 \cdot 50 + 765 \cdot 30 + 500 \cdot 20 = \$67,950.$$

- What is her total revenue?

$$\text{Revenue} = (\text{Rev per acre corn})\cdot(\text{acres of corn}) + (\text{Rev per acre wheat})\cdot(\text{acres of wheat}) + (\text{Rev per acre soy})\cdot(\text{acres of soy})$$

$$= 651 \cdot 50 + 523 \cdot 30 + 543 \cdot 20 = \$59,100.$$

Matrix Multiplication: A motivating example (cont)

- She encodes costs and revenues in a 2×3 matrix:

$$\begin{bmatrix} \$700 \text{ per acre corn} & \$765 \text{ per acre wheat} & \$500 \text{ per acre soy} \\ \$651 \text{ per acre corn} & \$523 \text{ per acre wheat} & \$543 \text{ per acre soy} \end{bmatrix}$$

Note: we usually don't include units of measure inside matrices. I'm doing this now to help keep track of what various entries mean.

- She encodes land usage in a 3×1 column matrix:

$$\begin{bmatrix} 50 \text{ acres corn} \\ 30 \text{ acres wheat} \\ 20 \text{ acres soy} \end{bmatrix}$$

- The "correct" definition of matrix multiplication computes the total cost and total revenue

$$\begin{bmatrix} 700 & 765 & 500 \\ 651 & 523 & 543 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix} = \begin{bmatrix} 700 \cdot 50 + 765 \cdot 30 + 500 \cdot 20 \\ 651 \cdot 50 + 523 \cdot 30 + 543 \cdot 20 \end{bmatrix} = \begin{bmatrix} 67,950 \\ 59,100 \end{bmatrix}$$

Tan, Chapter 2.5, Exercise 43 I

William's and Michael's stock holdings are given by the matrix

$$A = \begin{bmatrix} 200 & 300 & 100 & 200 \\ 100 & 200 & 400 & 0 \end{bmatrix}$$

where the rows record William's, then Michael's holdings, and the columns are holdings in the stocks BAC, GM, IBM, and TRW, respectively.
At the close of trading on a certain day, the prices (in dollars per share) of the stocks are given by the matrix

$$B = \begin{bmatrix} 54 \\ 48 \\ 98 \\ 82 \end{bmatrix}$$

(a.) Find AB.
(b.) Explain the meaning of the entries in the matrix AB.

Kaitlin and her friend Emma have returned to the U.S. from a tour of four cities: Oslo, Stockholm, Copenhagen, and Saint Petersburg. They now wish to exchange the various foreign currencies that they have accumulated for U.S. dollars (USD). Kaitlin has 82 Norwegian krones, 68 Swedish krones, 62 Danish krones, and 1200 Russian rubles. Emma has 64 Norwegian krones, 74 Swedish krones, 44 Danish krones, and 1600 Russian rubles. Suppose the exchange rates are 0.1651 USD for one Norwegian krone, 0.1462 USD for one Swedish krone, 0.1811 USD for one Danish krone, and 0.0387 for one Russian ruble.

(a.) Write a 2×4 matrix A giving the value of the various currencies held by Kaitlin and Emma. (Note: the answer is not unique!)

(b.) Write a column matrix B giving the exchange rates for the various currencies. (Order the entries so that the product AB will make sense)

(c.) If Kaitlin and Emma exchange all their foreign currencies for U.S. dollars, how much will they each have (in USD)?

The Cinema Center consists of four theaters, Cinemas I, II, III, and IV. The admission price for one feature at the center in \$4 for children, \$6 for students, and \$8 for adults. The attendance for the Sunday matinee is given by the matrix

$$A = \begin{bmatrix} 225 & 110 & 50 \\ 75 & 180 & 225 \\ 280 & 85 & 110 \\ 0 & 250 & 225 \end{bmatrix}$$

where the rows represent Cinemas I, II, III, and IV, respectively, and the columns represent children, students, and adults, respectively.

(a.) Write a column vector B representing the admission prices. (Order the entries so that the product AB will make sense.)

(b.) Compute AB, the column vector showing the gross receipts for each theater.

Tan, Chapter 2.5, Exercise 49 II

(c.) Find the total revenue collected at the Cinema Center from admissions on that day.

Cindy regularly makes long-distance phone calls to three foreign cities: London, Hong Kong, and Tokyo. The matrices A and B give the lengths (in minutes) of her calls during peak and non-peak hours, respectively, to each of these three cities during the month of June:

$$A = \begin{bmatrix} 80 & 60 & 40 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 300 & 150 & 250 \end{bmatrix}$$

where the columns represent London, Hong Kong, and Tokyo, respectively. The cost of the calls (in dollars per minute) for the peak and non-peak periods in the month in question are given, respectively, by the matrices:

$$C = \begin{bmatrix} 0.34 \\ 0.42 \\ 0.48 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0.24 \\ 0.31 \\ 0.35 \end{bmatrix}$$

Tan, Chapter 2.5, Exercise 55 II

(a.) Compute the matrix AC and explain what it represents.

(b.) Compute the matrix BD and explain what it represents.

(c.) Compute the matrix $AC + BD$ and explain what it represents.

(d.) Compute the matrix $A + B$ and explain what it represents.

(e.) Compute the matrix $(A + B)(C + D)$. Notice the result is different from the answer you obtained for $AC + BD$. Using the interpretations you developed in the previous parts, explain why $(A + B)(C + D)$ and $AC + BD$ should not necessarily be equal.