MA162: Finite mathematics

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Schedule:

- Seventh Web Assign assignment (Chapter 2.5, 2.6) due on Friday, Septemeber 27 by 6:00 pm.
- Exam 1 on Monday, September 30 from 5:00 pm to 7:00 pm

Today we cover Chapter 2.6: Matrix Inverses.

Matrix Inverses

- Consider the system of equations

$$\begin{array}{rcrcl} x & + & 2y & = & 2 \\ 2x & - & y & = & 4 \end{array}$$

- Using matrix multiplication notation, we can write this as

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- Perhaps we can solve for $\begin{bmatrix} x \\ y \end{bmatrix}$ by dividing?

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \div \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Matrix Inverses (cont)

- There is no well defined notion of *matrix division*, but something similar will work.

- Given matrices A, X, B, so that

$$A \cdot X = B$$

  we can try to solve for X by multiplying (on the left!) by *the inverse of* A, A^{-1},

$$X = A^{-1} \cdot B$$

- Limitations to this idea:

  - Some matrices don't have an inverse

  - Even if a matrix has an inverse, it might not be easy to compute

The Identity Matrix

- The n dimensional identity matrix is an has 1's along its main diagonal and 0's everywhere else. It is denoted I_n.
- $$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
- $$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- $$I_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Identity Matrix (cont)

- Even though there are infinitely many identity matrices (one for each dimension), we will often just write I, without the dimensional subscript.

- Identity matrices are always square. There is no 2×5 identity matrix, for instance.

- Multiplying a square matrix by its corresponding identity matrix doesn't do anything:

$$\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 5 \cdot 0 & 3 \cdot 0 + 5 \cdot 1 \\ 4 \cdot 1 + 2 \cdot 0 & 4 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$$

- Thus, the identity matrix is to matrix multiplication as the number 1 is to regular multiplication.

What is an Inverse Matrix?

- The multiplicative inverse of a real number a is the number a^{-1} with the property that

$$a \cdot a^{-1} = 1$$

- Analogously, the inverse of a matrix A is the matrix A^{-1} with the property that

$$A \cdot A^{-1} = I \qquad \text{and} \qquad A^{-1} \cdot A = I$$

Not all matrices have an inverse

- Does $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ have an inverse?

- Suppose it does, and lets write the inverse as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and use

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  to solve for the unknown entries.

Systematic Method for Finding Matrix Inverse

- We take another look at the matrix from the first slide:

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

- To find the inverse (or to deteremine none exists), create the augmented matrix:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right]$$

- Now do Gauss-Jordan Elimination. If you can get the identity matrix on the left side, then what appears on the right side will be the inverse matrix.
- If you can't get the identity matrix on the left side (perhaps an entire row on left side becomes zero), then the matrix does not have an inverse.

Formula for Inverse of 2×2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

- A^{-1} exists if and only if $ad - bc \neq 0$
- $$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$
- Previous Exmple: $ad - bc = 1 \cdot (-1) - 2 \cdot 2 = -5 \neq 0$, so this has an inverse. The inverse is

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-1}{-5} & \frac{-2}{-5} \\ \frac{-2}{-5} & \frac{1}{-5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

Computing an inverse

Find the inverse (if it exists)

$$\begin{bmatrix} 2 & -3 & -4 \\ 0 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Computing another inverse

Find the inverse (if it exists)

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & -1 & -1 & 3 \end{bmatrix}$$

Bob, a nutritionist who works for the University Medical Center, has been asked to prepare special diets for two patients, Susan and Tom. Bob has decided that Susan's meals should contain at least 400 mg of calcium, 20 mg of iron, and 50 mg of vitamin C, whereas Tom's meals should contain at least 350 mg of calcium, 15 mg of iron, and 40 mg of vitamin C. Bob has also decided that the meals are to be prepared from three basic foods: Food A, Food B, and Food C. The special nutritional contents of these foods are summarized in the accompanying table. Find how many ounces of each type of food should be used in a meal so that the minimum requirements of calcium, iron, and vitamin C are met for each patient's meals.

	Calcium	Iron	Vitamin C
Food A	30	1	2
Food B	25	1	5
Food C	20	2	4

Lawnco produces three grades of fertilizers. A 100-lb bag of grade A contains 18 lb of nitrogen, 4 lb phosphate, and 5 lb of potassium. A 100-lb bag of grade B contains 20 lb of nitrogen, 4 lb phosphate, and 4 lb of potassium. A 100-lb bag of grade C contains 24 lb of nitrogen, 3 lb phosphate, and 6 lb of potassium. How many 100-lb bags of each of the three grades of fertilizer should Lawnco produce if

(a.) $26,400$ lb of nitrogen, 4900 lb of phosphate, and 6200 lb of potassium are available and all the nutrients are used?

(b.) $21,800$ lb of nitrogen, 4200 lb of phosphate, and 5300 lb of potassium are available and all the nutrients are used?