Research Statement

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My research area is algebraic combinatorics. I have focused on two problems: studying the Rees product of the cubical lattice with the chain and studying inequalities of descent words in Dowling lattices. For the former we try to understand this Rees product in terms of its Möbius function, its order complex and top homology group, and its representation over the symmetric group. The current approach to the latter, is to apply boustrophedon transformations and determinants.

1 The Rees Product of the Cubical Lattice with the Chain

The Rees product on posets is derived from the Rees algebra studied in commutative algebra. Björner and Welker introduced the following definition.

Definition 1.1 *For two graded posets P and Q with rank function ρ the* Rees product*, denoted $P \star Q$, is the set of ordered pairs (p, q) in the Cartesian product $P \times Q$ with $\rho(p) \geq \rho(q)$. These pairs are partially ordered by $(p, q) \leq (p', q')$ if $p \leq_P p'$, $q \leq_Q q'$, and $\rho(p') - \rho(p) \geq \rho(q') - \rho(q)$.*

The Rees product of the Boolean algebra on n elements with the chain of appropriate length was shown by Jonnson to have Möbius value equal to the nth derangement number, up to a sign, settling a conjecture of Björner and Welker [1]. In recent work Shareshian and Wachs [4] have studied the poset homology of the order complex of this poset.

The next most natural poset to work with is to replace the Boolean algebra with the face lattice of the n-cube. The *Rees product of the cubical lattice with the chain* is the rank $n + 2$ poset

$$G_n = \big((\mathscr{C}_n \backslash \hat{0}) \star C_{n+1}\big) \cup (\hat{0}, \hat{1}),$$

where $\mathscr{C}_n$ is the n-cubical lattice, that is, the face lattice of the n-dimensional cube.

Theorem 1.2 *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu_{G_n}(\hat{0}, \hat{1}) \quad = \quad (-1)^n \cdot n \cdot \operatorname{per} \begin{bmatrix} 1 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & \cdots & 2 & 1 \end{bmatrix},$$

that is, n times the permanent of a square $(n-1) \times (n-1)$ matrix having 1's on the diagonal and 2's everywhere else.

We give a bijective proof of this theorem using skew diagrams associated to double augmented barred and signed permutations which are falling. The bijection is unusual in that we work with permutations in cycle notation. We also prove the following interesting result.

Theorem 1.3 *There exists a bijection between the set of all fixed point free permutations in the symmetric group on n elements and the set of all standard skew diagrams λ/μ having n boxes and hooks of size greater than 1.*

Next, we consider homological questions for the poset G_n. By a result of Björner and Welker [1], we know the order complex $\Delta(G_n)$ is a Cohen-Macaulay complex and has vanishing homology groups in every dimension except in top dimension. The order complex $\Delta(G_n)$ contains a set of subcomplexes, indexed by falling permutations. Each is isomorphic to the suspension of the barycentric subdivision of the boundary of the n-cube. We prove

Theorem 1.4 *The set $\{\rho_\sigma : \sigma \text{ is a falling permutation}\}$ forms a basis for $H(G_n)$ over $\mathbb{Z}$.*

Further, we can consider a representation of this top homology group $H(G_n)$ over the symmetric group.

Theorem 1.5 *There exists an S_n-module isomorphism between $H(\Delta(G_n))$ and $\oplus 2^{n-|\lambda_1|} S^\lambda$ where the direct sum is over all partitions λ with each λ_i shaped into hooks of certain shapes taken with multiplicity $2^{n-|\lambda_1|}$.*

2 Descent Words

A standard labeling for the maximal chains of the Boolean algebra on n elements assigns a permutation in S_n to a chain. Each permutation can be assigned an ab word $w = w_1 \cdots w_{n-1}$ where

$$w_i = \begin{cases} a & \text{if } \pi_i < \pi_{i+1} \\ b & \text{if } \pi_i > \pi_{i+1} \end{cases}$$

The descent statistic, $[w]$, is the number of permutations with word w. In the case of the Boolean algebra, the alternating word $[ababa\cdots] = [babab\cdots]$ maximized the descent statistic. This result has been proven independently by Niven, de Bruijn, and Viennot among others.

It is natural to consider which ab word maximizes the descent statistic for a S_n-labeling of the partition lattice. One way to count the ab words uses determinants. Niven states the following result for the Boolean algebra.

Theorem 2.1 (Niven) *The number of permutations with descent word w equals the determinant of order $r+1$ with the binomial coefficient $\binom{k_i}{k_{j-1}}$ at the intersection of the i-th row and j-th column $(i, j = 1, \ldots, r)$, where it is understood that $k_0 = 0, k_{r+1} = n$, and that $\binom{m}{s} = 0$ if $m < s$.*

We generalize this determinant beyond the poset B_n. Any poset whose lower order ideals are isomorphic for elements of the same rank or by duality whose upper order ideals are isomorphic for elements of the same rank can have its descent words counted by these determinants. In particular, the partition lattice has isomorphic upper order ideals of the same rank. The number of elements of rank i in Π_{n+1} is equal to the Stirling number $s(n, n-i)$.

Another way to study this descent statistic is an operation on sequences called the *boustrophedon* or "ox-plowing" transformation introduced by Millar, Sloane, and Young. The boustrophedon operation generates a triangular array and can be used to easily calculate the number of ab words which arise from labeled chains in the Boolean algebra.

We generalize this to a labeling of the partition lattice. For any word w, define a triangular array $\{t_{n,i}\}$ where $t_{1,1} = 0$ and

$$t_{n,i} = \begin{cases} i \sum_{j=i+1}^{n-1} t_{n-1,j} & \text{if } w_1 = a \\ i \sum_{j=1}^{i} t_{n-1,j} & \text{if } w_1 = b \end{cases}$$

Theorem 2.2 *For a given w of length $n-1$, $\sum_{i=1}^{n} t_{n,i}$ equals $[w]$, the number of maximal chains in Π_{n+2} whose labels have word w.*

Furthermore, if we modify this transformation we can count the number of chains with word w in $L_{n,k}$ the kth Dowling lattice. This time we define the array $\{t^k_{n,i}\}$ by $t^k_{1,1} = 1$ and

$$t^k_{n,i} = \begin{cases} (k(i-1)+1)\sum_{j=i+1}^{n-1} t^k_{n-1,j} & \text{if } w_1 = a \\ (k(i-1)+1)\sum_{j=1}^{i} t^k_{n-1,j} & \text{if } w_1 = b \end{cases}$$

When $k = 1$, notice we are back in the case of the partition lattice.

Calculations and partial results give the following:

Conjecture 2.3 *Let m be the length of an ab word. For m odd, the word $baba\cdots bab$ maximized the descent statistic. For m even, the word $baba\cdots babb$ maximized the descent statistic.*

3 Future Research

Several questions have arisen from the research on Rees products and the cubical lattice.

1. The Möbius values of the Rees product of the Boolean algebra and the chain and the cubical lattice and the chain are equal to a coefficient multiplied by the permanent of a matrix with one number on the diagonal and another number in every other position. Given two numbers, p and q, is it possible to find a poset whose Möbius value corresponds to the permanant of

$$\text{per} \begin{bmatrix} s & r & r & \cdots & r \\ r & s & r & \cdots & r \\ r & r & s & \cdots & r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & s \end{bmatrix} ?$$

2. Another research direction would be to consider the Rees product of the partition lattice with the chain. After we understand the Rees products of face lattices of a polytopes with chains, it is natural to look at the product with a lattice that does not correspond to a polytope. It is not too difficult to give a formula using multiple summations to define the Möbius function of this product, but does there exist a nicer closed formula?

3. It would be interesting to consider what kind of discrete Morse function could be placed on Rees products. If two posets have discrete Morse matchings, is there a canonical way to describe a discrete Morse matching on the Rees product?

There are also problems which have come up relating to the current work on descent words.

4. One extension of the boustrophedon results was into Dowling lattices, but there is another direction to consider. Ehrenborg and Mahajan extended original results on inequalities for *ab* words in the Boolean algebra to its q-analogue, the subspace lattice. Hanlon, Hersh, and Shareshian introduce a q-analogue to the partition lattice. Can we find similar q inequalities in this analogue?

5. Ehrenborg, Levin, and Readdy use a probabilistic approach to study the descent statistic. Is there a meaningful way to apply this technique to the weighted chains in the Dowling lattices?

6. We know there is boustrophedon transformation on the vector $(1, 0, 0, \ldots)$ which calculates the number of ab words for the Boolean algebra, the partition lattice, and the Dowling lattices. For any similar transformation on this vector does there exist a poset whose words can be counted by the transformation? What characteristics does the transformation have to have?

Finally, I have a few other ideas which I am interested in pursuing.

7. Stanley studies h-vectors and local h-vectors in [5]. He shows the local h-vector of the barycentric subdivision of the simplex is found using excedances in fixed point free permutations. Is there a similar result for the local h-vector of the barycentric subdivision of the cube? There exists sets of signed permutations whose excedances seem to describe the local h-vector. What is the signed analog for these fixed-point free permutations? Further, what is the "derangement complex", a set of cells in the barycentric subdivision of a simplex or a cube corresponding to fixed-point free permutations? Is this thing connected? What is its homology?

8. Another tough question concerns rank-selected subposets. It has been shown that a rank-selected subposet of a poset with a CL-labeling also has a CL-labeling. This result can be proved by using a recursive atom ordering on the poset. The equivalent result for EL-labeling is not known. Is there a different recursive technique that could be used in the case of EL-labelings? Can we find a poset property that is equivalent to an EL-labeling and use it in the rank-selected case?

4 Undergraduate Research

Research projects are very good way for undergraduates to expand their knowledge and challenge themselves. A project allows a student to experience the scientific method the way a mathematician does, from the excitement of finding an interesting question, to the frustration that comes as they work on the problem, ending with the joy of finding a solution.

Combinatorics offers many topics suitable for undergraduates. For example, many of the questions in combinatorics can be phrased very simply. One can ask questions like "How many colors does it take to color a given map so that no two adjacent countries have the same color?" or "What is the alternating sum of faces in an n-cube?" or "What are the possible values for an n by n magic square?" The questions are simple although the solutions may be very difficult. These simply phrased questions also lead students deeper into more technical questions as the learn more techniques to make their problem solving easier. Hopefully, they will enjoy their sucess with a topic and want to do more.

Enumeration questions also introduce students to structures which can often be counted in many different ways. The Catalan numbers have at least 66 interpretations and certainly more will be found. A student can choose their own method to solve the problems such that two student could have very different answers but still both be correct. One of the reasons I enjoy this subject is the freedom to be creative and an acceptance of multiple solutions.

I look forward to being a mentor for students interested in research, encouraging them to attend conferences, and encouraging them to discuss their work with others. This can lead to a self-confidence for the students that will help them whether they continue in mathematics or not.

References

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