Answers Test 2 Spring 2012

1. (a) Find the Taylor series expansion of $1/x$ about $x = 3$. Your answer should be in the form of an infinite series in which the general term is clearly indicated.

$$f(x) = \frac{1}{x} \xrightarrow{x=3} \frac{1}{3}$$

$$f'(x) = -\frac{1}{x^2} \longrightarrow -\frac{1}{3^2}$$

$$f''(x) = \frac{2!}{x^3}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \longrightarrow \frac{(-1)^n n!}{3^{n+1}}$$

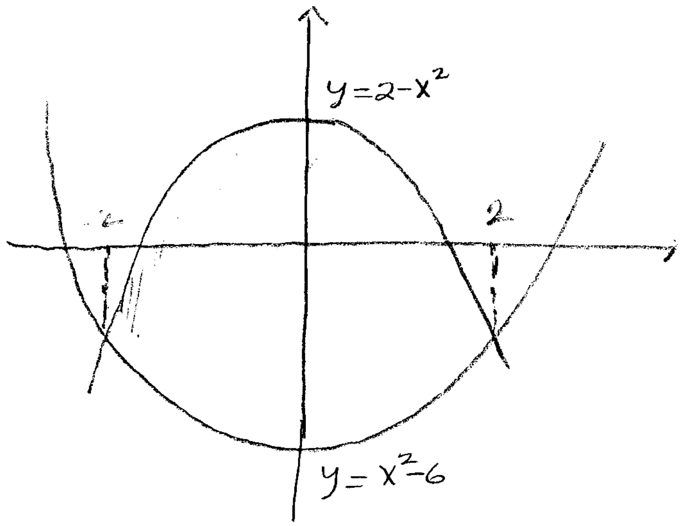
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{\frac{(-1)^n n!}{3^{n+1}}}{n!}(x-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x-3)^n$$

(b) Use your answer to (a) to write down the Taylor polynomial of degree two, $T_2(x)$, relative to $a = 3$ and the function $1/x$.

$$T_2(x) = \frac{1}{3} - \frac{1}{9}(x-3) + \frac{1}{27}(x-3)^2$$

2. Given the region R in the xy plane bounded by the graphs of $y = 2 - x^2$ and $y = -6 + x^2$. Find the area of R.



$$A = 2\int_0^2 \left[(2-x^2) - (x^2-6)\right] dx$$

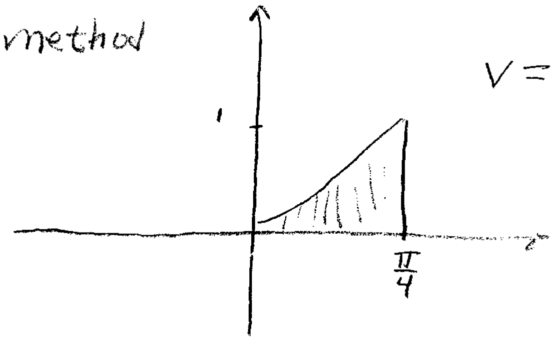
$$= 2\int_0^2 (8-2x^2)\,dx = 2\left(8x - \tfrac{2}{3}x^3\right)\Big|_0^2$$

$$= 2\left(16 - \tfrac{16}{3}\right) = \frac{64}{3}$$

3. Given the region, D, bounded by the graph of $y = \sqrt{\sin(2x)}, 0 \le x \le \pi/4$, and the lines $y = 0, x = \pi/4$.

(a) Find the volume obtained from rotating D about the x axis. You may express your answer in terms of π.

Washer method



$$V = \pi \int_0^{\frac{\pi}{4}} y^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \sin(2x) \, dx$$

$$= -\frac{\pi}{2} \cos(2x) \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2}$$

(b) Set up, <u>do not evaluate</u> an integral for the volume obtained from rotating D about the y axis.

Shell method

$$V = 2\pi \int_0^{\frac{\pi}{4}} x \, y \, dx$$

$$= 2\pi \int_0^{\frac{\pi}{4}} x \sqrt{\sin(2x)} \, dx$$

4.(a) A uniform cable hanging over the edge of a tall building is 40 feet long and weighs 60 lbs. How much work is required to pull the cable to the top of the building ?

$$\rho = \text{density} = \frac{60}{40} = \frac{3}{2}\ \text{lb/ft}$$

$$W = \int_0^{40} x\ (\rho\, dx) = \frac{3}{2}\int_0^{40} x\, dx$$

$$= \frac{3}{4}x^2\Big|_0^{40} = \frac{3}{4}(1600) = 1200\ \text{ft-lb}$$

(b) How much work is required to pull just 10 feet of the cable to the top of the building?

The other 30 ft will be raised 10ft

Total work is

$$W = \underbrace{\frac{3}{2}\int_0^{10} x\, dx}_{\text{First 10 ft}} + \underbrace{\overset{\text{ft}}{10}\,\Big(\overset{45\ \text{lb}}{\tfrac{3}{2}\cdot 30}\Big)}_{\text{Last 30 ft}}$$

$$= \frac{3}{4}x^2\Big|_0^{10} + 450$$

$$= 75 + 450 = 525\ \text{ft-lb}$$

5. Find the following integrals. Indicate clearly which integration method you are using. For example if integration by parts is used, write: let $u = \ldots$, $dv = \ldots$

(a) $\int_0^\pi \sin^3 x\, dx$ $= \int_0^\pi \sin^2(x)\,(\sin(x)\,dx) = \int_0^\pi (1-\cos^2(x))(\sin(x)\,dx)$

$u = \cos(x)$ — Sub. method

$du = -\sin(x)\,dx$

$= -\int_1^{-1} (1-u^2)\,(du)$

$= \int_{-1}^{1} (1-u^2)\,du = 2\int_0^1 (1-u^2)\,du$

$= 2\left(u - \frac{u^3}{3}\right) = \frac{4}{3}$

$\int u\,dv$

(b) $\int x^4 \ln x\, dx$ $= \underbrace{\frac{x^5 \ln x}{5}}_{uv} - \underbrace{\int \frac{x^5}{5}\left(\frac{dx}{x}\right)}_{\int v\,du}$

use parts

$u = \ln x$, $dv = x^4\,dx$

$du = \frac{dx}{x}$, $v = \frac{x^5}{5}$

$= \frac{x^5 \ln x}{5} - \frac{1}{5}\int x^4\,dx + C$

$= \frac{x^5 \ln x}{5} - \frac{1}{25}x^5 + C$

6. Find the following integrals. Indicate clearly which integration technique you are using.

(a) $\int \sqrt{25 - x^2}\, dx$

$$= \int 5\cos(\theta)(5\cos(\theta))\,d\theta$$

$$= \frac{25}{2}\int (1+\cos(2\theta))\,d\theta$$

$$= \frac{25}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

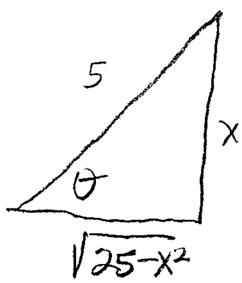
$$= \frac{25}{2}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$

$$= \frac{25}{2}\left(\sin^{-1}\left(\frac{x}{5}\right) + \frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5}\right) + C$$

Trig. Sub

$x = 5\sin(\theta)$

$dx = 5\cos(\theta)\,d\theta$



(b) $\int x\tan^3(x^2)\sec(x^2)dx$

$$\sec^2(x^2) - 1$$

$$= \int \tan^2(x^2)\,(x\sec(x^2)\tan(x^2)\,dx)$$

$$= \int (u^2-1)\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\left(\frac{u^3}{3} - u\right) + C$$

$$= \frac{1}{2}\left(\frac{\sec^3(x^2)}{3} - \sec(x^2)\right) + C$$

Trig sub.

$u = \sec(x^2)$

$du = 2x\sec(x^2)\tan(x^2)$