MA 114 — Calculus II
Exam III

Spring 2011
April 12, 2011

This is a closed book exam. There are seven (7) problems on eight (8) pages (including this cover page). Check and be sure that you have a complete exam.

No books or notes may be used during the exam. You may use a graphing calculator provided that it does not have symbolic manipulation capabilities. In addition, *any* device capable of electronic communication (cell phone, pager, etc.) must be turned off and out of sight during the exam.

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. If you need more space then use the backs of the exam pages.

Show your work. Answers without justification will receive no credit. Partial credit for a problem will be given only when there is coherent written evidence that you have solved part of the problem. In particular, answers that are obtained simply as the output of calculator routines will receive no credit. Finally, be aware that it is not the responsibility of the grader to determine which part of your response is to be graded. Be sure to erase or mark out any work that you do not want graded.

Name: ______________________________

Section: __________

Last four digits of student identification number: __________

Problem	Score	Total
1		15
2		15
3		15
4		15
5		10
6		14
7		16
		100

1) (15 pts) This problem considers the curve $y = \sqrt{4-x^2}$, $0 \le x \le \sqrt{3}$.

(a) (12 pts). Use the arc length formula to find the length of the curve.

If $y = \sqrt{4-x^2} = (4-x^2)^{\frac{1}{2}}$, then $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$

and $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4-x^2} = \frac{4}{4-x^2}$. So

$$L = \int_0^{\sqrt{3}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = 2\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = (2u = x,\ 2du = dx)$$

$$= 2\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = 2 \arcsin(u) \Big|_0^{\frac{\sqrt{3}}{2}} = 2\arcsin\left(\frac{\sqrt{3}}{2}\right) - 0$$

$$= \frac{2\pi}{3}$$

(b) (3 pts). Find the distance between the endpoints of the curve.

End points are $(0,0)$ and $(\sqrt{3}, 1)$

So distance $= \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$.

2) (15 pts) Determine whether each improper integral is convergent or divergent. If it is convergent, then evaluate it.

(a) (7 pts) $\int_1^2 \ln(x-1)\,dx = \lim_{t\to 1^+} \int_t^2 \ln(x-1)\,dx$

use IBP

$u = \ln(x-1)$, $dv = dx$

$du = \frac{dx}{x-1}$, $v = x-1$

$= \lim_{t\to 1^+}\left[(x-1)\ln(x-1)\Big|_t^2 - \int_t^2 dx\right]$

$= \lim_{t\to 1^+}\left[(x-1)\ln(x-1) - x\right]_t^2$

(see right hand side) $\longrightarrow$

$= \ln 1 - 2 - (-1) = \boxed{-1}$

- Now $\lim_{x\to 1^+}(x-1)\ln(x-1)$

$= \lim_{w\to 0^+} \frac{\ln(w)}{\frac{1}{w}} =$ (by L'Hospital's rule)

$= \lim_{w\to 0^+} \frac{\frac{1}{w}}{-\frac{1}{w^2}} = \lim_{w\to 0^+}(-w)$

$= 0.$

(b) (8 pts) $\int_0^\infty \frac{e^x}{e^{2x}+4}\,dx = \lim_{t\to\infty} \int_0^t \frac{e^x\,dx}{e^{2x}+4}$

let $2u = e^x$

$2\,du = e^x\,dx$,

$x=0$, $u=\frac{1}{2}$

$x=t$, $u=\frac{1}{2}e^t$

$= \lim_{t\to\infty} \int_{\frac{1}{2}}^{\frac{1}{2}e^t} \frac{2\,du}{4u^2+4}$

$= \frac{1}{2}\lim_{t\to\infty} \int_0^{\frac{1}{2}e^t} \frac{du}{u^2+1} = \frac{1}{2}\lim_{t\to\infty} \arctan u \Big|_{\frac{1}{2}}^{\frac{1}{2}e^t}$

$= \frac{1}{2}\lim_{t\to\infty} \arctan\left(\frac{e^t}{2}\right) - \frac{1}{2}\arctan\left(\frac{1}{2}\right)$

$= \frac{\pi}{4} - \frac{1}{2}\arctan\left(\frac{1}{2}\right)$

3) (15 pts) Consider the infinite series $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^3}$.

(a) (9 pts) Use the integral test to show that the series converges. You need to verify that your choice of f satisfies the hypotheses of this test.

Let $f(x) = \frac{1}{x(\ln x)^3}$. Then x and $(\ln x)^3$ are positive increasing functions on $[2,\infty)$ so $x(\ln x)^3$ is increasing and $f(x) = \frac{1}{x(\ln x)^3}$ is decreasing on $[2,\infty)$. Also f is continuous on this interval. Thus the hypotheses of the integral test are satisfied so $\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^3}$ converges if and only if

$$\infty > \int_2^{\infty} \frac{2\,dx}{x(\ln x)^3} = \lim_{t\to\infty} \int_2^t \frac{2\,dx}{x(\ln x)^3} = \left(u = \ln x,\ du = \frac{dx}{x}\right) = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{2\,du}{u^3}$$

$$= \lim_{t\to\infty} -u^{-2} \Big|_{\ln 2}^{\ln t} = \frac{1}{(\ln 2)^2}.$$

(b) (6 pts) The remainder estimate for the integral test states that under the hypotheses of the integral test, the Remainder $R_n = s - s_n$ satisfies

$$0 \le R_n \le \int_n^{\infty} f(x)\,dx.$$

Use this inequality to determine the least integer for which $R_n \le 0.01$

(from the above calculation)

$$R_n \le \int_n^{\infty} \frac{2\,dx}{x(\ln x)^3} = \lim_{t\to\infty} \left(-\frac{1}{(\ln x)^2}\right)\Big|_n^t = \frac{1}{(\ln n)^2}$$

So we want $\frac{1}{(\ln n)^2} \le \frac{1}{100}$

or $(\ln n)^2 \ge 100$, $\ln n \ge 10$, $n \ge e^{10}$.

Take n to be the least positive integer $\ge e^{10}$.

4) (15 pts) Approximate the integral $I = \int_0^1 e^{x^2}\,dx$ by the Midpoint rule, the Trapezoidal rule, and Simpson's rule using the specified number of subintervals.

In your answers you do not need to evaluate expressions of the form e^r.
For instance if the question asked for L_2 then, along with supporting work, you could leave the final answer in the form "$\frac{1}{2}e^0 + \frac{1}{2}e^{\frac{1}{4}}$" or "$\frac{1}{2}(e^0 + e^{\frac{1}{4}})$", etc.

(a) (5 pts) Calculate the Midpoint rule estimate, M_3, for I (note: 3 sub-intervals).

$$M_3 = \frac{1}{3}\left(e^{1/36} + e^{1/4} + e^{25/36}\right)$$

$0 \quad \frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{5}{6} \quad 1$

(b) (5 pts) Calculate the Trapezoidal rule estimate, T_3, for I (note: 3 sub-intervals).

$$T_3 = \frac{1}{6}\left(1 + 2e^{1/9} + 2e^{4/9} + e\right)$$

(c) (5 pts) Calculate the Simpson's rule estimate, S_4, for I (note: 4 sub-intervals).

$0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1$

$$S_4 = \frac{1}{12}\left[1 + 4e^{1/16} + 2e^{1/4} + 4e^{9/16} + e\right]$$

5) (10 pts) How large should we take n in order to guarantee that the Midpoint rule approximation for $\int_1^3 \frac{1}{x}\,dx$ is accurate to within 10^{-2}. **Recall that the error bound for the Midpoint rule for** $\int_a^b f(x)\,dx$ **is given by**

$$\left|E_M(n)\right| \le \frac{K(b-a)^3}{24n^2}, \text{ with } K = \max\left\{\left|f''(x)\right| : a \le x \le b\right\}.$$

Let $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$

and $|f''(x)| = \frac{2}{|x|^3} \le 2 = |f''(1)|$ on $[1, 3]$.

so $K = 2$ in the above inequality, $b-a=2$

and we want

$$|E_M| \le \frac{2 \cdot 2^3}{24n^2} = \frac{2}{3n^2} \le \frac{1}{100}$$

or $n^2 \ge \frac{200}{3}$, $n \ge \sqrt{\frac{200}{3}}$. Take n to be the first positive integer $\ge \sqrt{\frac{200}{3}}$

$= 9$

6) (14 pts)

(a) (6 pts) Find the tangent to the cycloid, $x = 2(\theta - \sin\theta),\ y = 2(1 - \cos\theta)$, at $\theta = \dfrac{\pi}{4}$.

At $\theta = \frac{\pi}{4}$, $x = 2\left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right)$, $y = 2\left(1 - \frac{\sqrt{2}}{2}\right)$

$$\frac{dy}{dx} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\frac{\pi}{4}} = \left.\frac{2\sin\theta}{2-2\cos\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{\sqrt{2}}{2-\sqrt{2}}$$

So tangent line equation is

$$y - (2-\sqrt{2}) = \frac{\sqrt{2}}{(2-\sqrt{2})}\left[x - \left(\frac{\pi}{2} - \sqrt{2}\right)\right]$$

(b) (8 pts) Find the length of the curve $x = 1 + 3t^2,\ y = 4 + 2t^3,\ 0 \le t \le 1$.

$$\frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = 6t^2, \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36t^2(1+t^2)$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 6t\sqrt{1+t^2}$$

$$L = 6\int_0^1 t\sqrt{1+t^2}\,dt \qquad u = 1+t^2, \quad du = 2t\,dt$$

$$= 2(1+t^2)^{\frac{3}{2}}\Big|_0^1 = 2\left(2^{\frac{3}{2}} - 1\right)$$

7) (16 pts) This problem considers partial fraction for rational functions and their integrations.

(a) (6 pts) Write out the form of the partial fraction decomposition of the function $\frac{2x+1}{(x^3-x^2)(x^2+1)}$. Don't determine the numerical values of the coefficients.

Factor as

$$\frac{2x+1}{x^2(x-1)(x^2+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$$

(b) (10 pts) Use the partial fraction method to evaluate the integral

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$$

$\frac{4y^2-7y-12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3}$. Multiply both sides by $y(y+2)(y-3)$ to get

$4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$

plug in $y=0$, get $-12 = -6A$, $A=2$,

plug in $y=-2$, get $18 = 10B$, $B = 18/10 = 9/5$

plug in $y=3$, get $3 = 15C$, $C = \frac{1}{5}$

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}dy = 2\int_1^2 \frac{dy}{y} + \frac{9}{5}\int_1^2 \frac{dy}{y+2} + \frac{1}{5}\int_1^2 \frac{dy}{y-3}$$

$$= 2\ln y + \frac{9}{5}\ln(y+2) + \frac{1}{5}\ln|y-3|\Big)_1^2$$

$$= 2\ln 2 + \frac{9}{5}\ln\left(\frac{4}{3}\right) - \frac{1}{5}\ln 2.$$