MA 114 Calculus II — Midterm Exam 1

NAME: SECTION:

LAST FOUR DIGITS OF STUDENT ID:

NOTES:

1. This is a closed book exam. There are six (6) problems on eight (8) pages. Check and be sure that you have a complete exam.
2. You may use a graphing calculator that does not have symbolic manipulation capabilities. Any device capable of electronic communication (cell phone, pager, etc.) must be turned off and out of sight during the exam.
3. Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. Please erase or mark out any work that you do not want graded. If you need more space, use the backs of the exam pages.
4. Unless specified otherwise, **show your work**; answers without any justification will receive no credit.

Problem	Score	Points
1		10
2-a		10
2-b		8
2-c		8
2-d		8
3		14
4		18
5		14
6		10
Total		100

1. Consider the geometric series $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \cdots\cdots$.

(a) Write the series in summation notation.

$$\sum_{n=1}^{\infty} 2(-\frac{2}{3})^{n-1}$$

(b) Does the series converge? If yes, find the sum of this series.

Yes.

$$\sum_{n=1}^{\infty} 2(-\frac{2}{3})^{n-1} = \frac{2}{1 - (-\frac{2}{3})} = \frac{6}{5}$$

2. For each of the following series, determine whether it converges or diverges. Show your justifications and state the name of the convergence test you use.

(a) $\sum_{n=1}^{\infty} \frac{2n^2 + 3^n}{-3 + 2(5^n)}$

First

$$\frac{\frac{2n^2+3^n}{-3+2(5^n)}}{\frac{3^n}{5^n}} = \frac{2\frac{n^2}{3^n} + 1}{-3\frac{1}{5^n} + 2} \to \frac{1}{2} < \infty$$

where

$$\lim_{n\to\infty} \frac{n^2}{3^n} = \lim_{x\to\infty} \frac{x^2}{3^x} = \lim_{x\to\infty} \frac{2x}{3^x ln3} = \lim_{x\to\infty} \frac{2}{3^x ln3 ln3} = 0$$

Because $\sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} (\frac{3}{5})^n$ is convergent, the series is convergent by the limit comparison test.

(b) $\sum_{n=1}^{\infty} \frac{n^2 - 2n + 3}{n^3 - 4n + 4}$

Because

$$\frac{\frac{n^2-2n+3}{n^3-4n+4}}{\frac{1}{n}} = \frac{1 - 2\frac{1}{n} + 3\frac{1}{n^2}}{1 - 4\frac{1}{n^2} + 4\frac{1}{n^3}} \to 1$$

and because $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the series is divergent by the limit comparison test.

2. (continued)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{n!}$

Because

$$\frac{\frac{(n+1)2^{n+1}}{(n+1)!}}{\frac{n2^n}{n!}} = \frac{2}{n} \to 0 < 1,$$

the series is convergent by the ratio test.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+2}{2n^2-n+5}\right)^n$

Because

$$\sqrt[n]{\left(\frac{n^2+2}{2n^2-n+5}\right)^n} = \frac{n^2+2}{2n^2-n+5} = \frac{1+2\frac{1}{n^2}}{2-\frac{1}{n}+5\frac{1}{n^2}} \to \frac{1}{2} < 1,$$

the series is convergent by the root test.

3. Show that the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{20n^2+1}$ is convergent. How many terms in the partial sum S_n do we need in order to find an approximate sum with error less than 0.01? Write down (but do not compute) this partial sum.

We apply the alternating series test:

1. $\lim_{n\to\infty} \frac{n}{20n^2+1} = \lim_{n\to\infty} \frac{1/n}{20+1/n^2} = 0.$

2. For $f(x) = \frac{x}{20x^2+1}$, $f'(x) = \frac{-20x^2+1}{(20x^2+1)^2} < 0$ for $x \geq 1$. Therefore $f(x)$ is decreasing for $x \geq 1$ and $f(n+1) < f(n)$. So $\{\frac{n}{20n^2+1}\}$ is decreasing.

Thus, the series is convergent by the alternating series test.

For $S = \sum_{n=1}^{\infty}(-1)^n \frac{n}{20n^2+1}$, we have $|S_n - S| < a_{n+1} = \frac{n+1}{20(n+1)^2+1}$. For the error to be less than 0.01, we find n such that $a_{n+1} \leq 0.01$.

$n = 3$: $a_4 = \frac{4}{320+1} > 0.01$

$n = 4$: $a_5 = \frac{5}{500+1} < 0.01$

(n can also be found by solving the inequality $\frac{n+1}{20(n+1)^2+1} < 0.01$.)

So we need four terms and $S_4 = -\frac{1}{21} + \frac{2}{81} - \frac{3}{181} + \frac{4}{321}$.

4. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-2)^n}{2n+1} x^n$. Show your justifications and be sure to check end points.

$$\left| \frac{\frac{(-2)^{n+1}}{2(n+1)+1} x^{(n+1)}}{\frac{(-2)^n}{2n+1} x^n} \right| = 2\frac{2n+1}{2n+3}|x| = 2\frac{2+1/n}{2+3/n}|x| \to 2|x|, \text{ as } n \to \infty.$$

Therefore the power series converges for $2|x| < 1$ or $|x| < \frac{1}{2}$. The radius of convergence is $\frac{1}{2}$.

To determine convergence interval, we check the end points:

- $x = -\frac{1}{2}$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{2n+1} x^n = \sum_{n=0}^{\infty} \frac{1}{2n+1}$. Because $\frac{1}{2n+1} > \frac{1}{2(n+1)}$ and because $\sum_{n=0}^{\infty} \frac{1}{2(n+1)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) is divergent, we conclude that $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ is divergent by the comparison test.

- $x = \frac{1}{2}$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{2n+1} x^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$. This is an alternating series. Since $\frac{1}{2(n+1)+1} < \frac{1}{2n+1}$, $\{\frac{1}{2n+1}\}$ is a decreasing sequence. Furthermore $\lim_{n\to\infty} \frac{1}{2n+1} = 0$. Therefore, $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ is convergent by the alternating series test.

The convergence interval is $(-\frac{1}{2}, \frac{1}{2}]$.

5. Find a power series representation for the function $f(x) = \dfrac{x}{(2-x^2)^2}$. Determine the radius of convergence.

Since $(\dfrac{1}{2-x^2})' = \dfrac{2x}{(2-x^2)^2}$, we have

$$\begin{aligned} \frac{x}{(2-x^2)^2} &= \frac{1}{2}(\frac{1}{2-x^2})' \\ &= \frac{1}{4}(\frac{1}{1-x^2/2})' \\ &= \frac{1}{4}(\sum_{n=0}^{\infty}(\frac{x^2}{2})^n)' \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\frac{2n}{2^n}x^{2n-1} \end{aligned}$$

Since $\sum_{n=0}^{\infty}(\frac{x^2}{2})^n$ converges for $\frac{x^2}{2} < 1$ or $|x| < \sqrt{2}$, its radius of convergence is $\sqrt{2}$. Since differentiation of a power series does not change the radius of convergence, the radius of convergence for the above power series is also $\sqrt{2}$.

6. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and let S be its sum. Let $S_n = \sum_{i=1}^{n} a_i$ be the n-th partial sum. Answer each of the following questions; **no justification** is needed.

(a) What is $\lim_{n\to\infty} S_n$?

S

(b)What is $\lim_{n\to\infty} a_n$?

0

(c) If $a_n > 0$ for all n, is $\{S_n\}$ an increasing or decreasing sequence? Find an upper bound and a lower bound for $\{S_n\}$, if it exists.

increasing. lower bound: 0; upper bound: S

(d) Let $\sum_{n=1}^{\infty} b_n$ be another series and let $T_n = \sum_{i=1}^{n} b_i$ be its partial sum. If $0 < b_n < 2a_n$ for all n, find an upper bound and a lower bound for $\{T_n\}$, if it exists.

lower bound: 0; upper bound: $2S$