1. (a) (6 pts) Write out the form of the partial fraction decomposition of the function $\frac{2x-3}{(x-1)(x^2+x+1)^2}$. Don't determine the numerical values of the coefficients.

$$\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

(b) (12 pts) Use the method of partial fractions to evaluate the integral

$$I = \int_0^1 \frac{2}{(x+1)(x^2+1)}dx.$$

$$\frac{2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$2 = A(x^2+1) + (Bx+C)(x+1) \quad \text{--- (1)}$$

Put $x=-1$. We get $2 = 2A \Rightarrow A = 1$

Comparing coefficients on both sides of (1), we obtain

$$A+B=0 \Rightarrow B=-1,$$
$$A+C=2 \Rightarrow C=1.$$

$$\int_0^1 \frac{1}{x+1}dx = \left[\ln|x+1|\right]_0^1 = \ln 2$$

$$-\int_0^1 \frac{x}{x^2+1}dx = -\left[\tfrac{1}{2}\ln(x^2+1)\right]_0^1 = -\tfrac{1}{2}\ln 2$$

$$\int_0^1 \frac{1}{x^2+1}dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$$

$$\therefore I = \tfrac{1}{2}\ln 2 + \frac{\pi}{4}.$$

2. Determine whether each improper integral below is convergent or divergent. If it is convergent, then evaluate it.

(a) (10 pts) $\int_1^\infty \frac{dx}{x(x+1)}$.

$$\int_1^\infty \frac{dx}{x(x+1)} = \lim_{t\to\infty} \int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \lim_{t\to\infty} \Big[\ln x - \ln(x+1)\Big]_1^t$$

$$= \lim_{t\to\infty} \left(\ln t - \ln(t+1)\right) + \ln 2$$

$$= \lim_{t\to\infty} \ln\left(\frac{t}{t+1}\right) + \ln 2$$

$$= \ln\left(\lim_{t\to\infty} \frac{t}{t+1}\right) + \ln 2$$

$$= \ln 1 + \ln 2$$

$$= \ln 2.$$

$\therefore$ The given improper integral is convergent.

(b) (10 pts) $\int_1^2 \frac{x}{\sqrt{x-1}}dx$.

$$I = \int_1^2 \frac{x}{\sqrt{x-1}}\,dx = \lim_{t\to 1^+} \int_t^2 \frac{x}{\sqrt{x-1}}\,dx.$$

Let $u = x-1$. Then $du = dx$, $x = 1+u$. When $x=t$, $u = t-1$. When $x=2$, $u=1$. Hence

$$I = \lim_{t\to 1^+} \int_{t-1}^1 \frac{1+u}{\sqrt{u}}\,du$$

$$= \lim_{t\to 1^+} \left(\int_{t-1}^1 u^{-\frac{1}{2}}\,du + \int_{t-1}^1 u^{\frac{1}{2}}\,du \right)$$

$$= \lim_{t\to 1^+} \left(\left[2u^{\frac{1}{2}}\right]_{t-1}^1 + \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{t-1}^1 \right)$$

$$= \lim_{t\to 1^+} \left(2(1-\sqrt{t-1}) + \frac{2}{3}(1-(t-1)^{3/2}) \right)$$

$$= 2 + \frac{2}{3} = \frac{8}{3}.$$

$\therefore$ The given improper integral is convergent.

3. (16 pts) Use the integral test to show that the infinite series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent. You need to verify that your choice of f satisfies the hypotheses of this test.

Let $f(x) = \frac{\ln x}{x^2}$. Then $f'(x) = \frac{1 - 2\ln x}{x^3}$.

Since $f(x) > 0$ and $f'(x) < 0$ on $[2, \infty)$, the function f satisfies the hypotheses of the integral test.

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left(\left[-\frac{\ln x}{x}\right]_1^t + \int_1^t \frac{1}{x}\cdot\frac{1}{x}\,dx \right)$$

$$= \lim_{t\to\infty} \left(-\frac{\ln t}{t} + \left[-\frac{1}{x}\right]_1^t \right)$$

$$= -\lim_{t\to\infty} \frac{\ln t}{t} - \lim_{t\to\infty} \frac{1}{t} + 1$$

$$= 0 + 0 + 1$$

$$= 1.$$

By the integral test, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is convergent.

4. (a) The following table gives the values of a function W at the given points in the interval $[0, H]$:

x	0	$H/4$	$H/2$	$3H/4$	H
$W(x)$	0.093	0.067	0.082	0.030	0.009

Let $I = \int_0^H W(x)dx$. In each part below, you may leave your answer as an expression that involves sums and products of numbers or decimals.

(i) (5 pts) Use the Trapezoidal Rule with $n = 4$ to obtain an approximate value of I expressed in terms of H.

$$T_4 = \frac{H}{8}(0.093 + 2\times 0.067 + 2\times 0.082 + 2\times 0.030 + 0.009)$$

$$= (0.0575)H$$

(ii) (5 pts) Use Simpson's Rule with $n = 4$ to obtain an approximate value of I expressed in terms of H.

$$S_4 = \frac{H}{12}(0.093 + 4\times 0.067 + 2\times 0.082 + 4\times 0.030 + 0.009)$$

$$= (0.0545)H$$

(iii) (2 pts) Use the approximate value of I obtained either in (i) or in (ii) to get an estimate for the average value (or integral average) of W on $[0, H]$.

$$\frac{T_4}{H} = 0.0575 \quad \text{or} \quad \frac{S_4}{H} = 0.0545$$

4. (b) (8 pts) How large should we take n in order to guarantee that the approximation by Simpson's Rule for $\int_1^2 \ln x\, dx$ is accurate to within 10^{-4}? Recall that the error bound for Simpson's rule for $\int_a^b f(x)dx$ is given by

$$|E_S(n)| \le \frac{K(b-a)^5}{180n^4}, \qquad \text{with } K = \max\left\{\left|f^{(4)}(x)\right| : a \le x \le b\right\}.$$

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2},$$

$$f'''(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = -\frac{6}{x^4}$$

$$K = \max\left\{\frac{6}{x^4} : 1 \le x \le 2\right\}$$

$$= 6$$

$$\frac{K(b-a)^5}{180n^4} < 10^{-4} \iff \frac{1}{30n^4} < 10^{-4}$$

$n > \sqrt[4]{\frac{1}{30}} \times 10 \approx 4.28$ and n is even.

$\therefore\ n = 6.$

5. (10 pts) Set up, but do not evaluate, an integral for the length of the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where } a > 0, \ b > 0.$$

The given curve can be described parametrically by

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi.$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

Alternate Solution

For the part of the given ellipse above the x-axis,

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

$$\frac{dy}{dx} = -\frac{b}{a^2} \frac{x}{\sqrt{1-\frac{x^2}{a^2}}}, \qquad \left(\frac{dy}{dx}\right)^2 = \frac{b^2x^2}{a^2(a^2-x^2)}$$

$$L = 2\int_{-a}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\int_{-a}^{a} \sqrt{1 + \frac{b^2x^2}{a^2(a^2-x^2)}}\, dx$$

$$= 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2-x^2)}}\, dx.$$

6. A curve C is defined by the parametric equations

$$x = t^3 - 3t, \qquad y = 3t^2 - 9.$$

(a) (10 pts) Find the Cartesian coordinates of the points on C where (i) the tangent is horizontal and (ii) the tangent is vertical.

$$\frac{dx}{dt} = 3t^2 - 3 = 3(t^2-1), \qquad \frac{dy}{dt} = 6t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{3(t^2-1)} = \frac{2t}{t^2-1}$$

$\frac{dy}{dt} = 0 \Leftrightarrow t = 0$. At $t=0$, $\frac{dx}{dt} \neq 0$

and $(x, y) = (0, -9)$.

$\therefore$ The tangent is horizontal at $(x, y) = (0, -9)$.

$\frac{dx}{dt} = 0 \Leftrightarrow t = \pm 1$. At $t = 1$, $\frac{dy}{dt} \neq 0$

and $(x, y) = (-2, -6)$. At $t = -1$, $\frac{dy}{dt} \neq 0$

and $(x, y) = (2, -6)$.

$\therefore$ The tangent is vertical at

$(x, y) = (-2, -6)$ and $(x, y) = (2, -6)$.

6. (b) (6 pts) Determine the values of t for which (i) the curve is concave upward and (ii) the curve is concave downward.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2t}{t^2-1}\right)}{3(t^2-1)}$$

$$= \frac{2}{3} \cdot \frac{-(1+t^2)}{(t^2-1)^3}$$

$$\therefore \quad \frac{d^2y}{dx^2} > 0 \quad \text{if} \quad t^2-1 < 0$$

$$\frac{d^2y}{dx^2} < 0 \quad \text{if} \quad t^2-1 > 0$$

The curve is concave upward when $-1 < t < 1$.

It is concave downward when $t > 1$ or $t < -1$.