MA 114 - - Calculus II
Final Exam

Spring 2011

This is a closed book exam. There are eight (8) problems on nine (9) pages (including this cover page). Check and be sure that you have a complete exam.

No books or notes may be used during the exam. You may use a graphing calculator *provided* that it does not have symbolic manipulation capabilities. In addition, *any device capable of electronic communication (cell phone, pager, etc.) must be turned off and out of sight during the exam.*

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. If you need more space then use the backs of the exam pages.

Show your work. Answers without justification will receive no credit. Partial credit for a problem will be given only when there is coherent written evidence that you have solved part of the problem. In particular, answers that are obtained simply as the output of calculator routines will receive no credit. Finally, be aware that it is not the responsibility of the grader to determine which part of your response is to be graded. Be sure to erase or mark out any work that you do not want graded, and also be sure to indicate if any part of your answer is on a different page.

Name: ______________________________

Section: __________

Last four digits of student identification number: __________

Problem	Score	Total
1		13
2		12
3		12
4		15
5		12
6		12
7		12
8		12
		100

1) In parts a and b below, you are asked to convert between rectangular and polar coordinates. **You should given an exact answer, not a decimal approximation.** As always, show your work!

(a) (4 pts) Convert from polar to rectangular coordinates: $(r, \theta) = (2, \pi/3)$.

$$x = 2\cos\left(\frac{\pi}{3}\right) = 1$$

$$y = 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}$$

(b) (4 pts) Convert from rectangular to polar coordinates: $(x, y) = (-\sqrt{3}, -1)$. **Be sure to find all sets of polar coordinates corresponding to the given Cartesian point.**

$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$ so

$-\sqrt{3} = 2\cos\theta$, $-1 = 2\sin\theta$, or $\sin\theta = -\frac{1}{2}$

and $\cos\theta = -\frac{\sqrt{3}}{2}$, so $\theta = \frac{7\pi}{6}$ works.

Polar coordinates are $\left(2, \frac{7\pi}{6} + 2k\pi\right)$, $k = 0, \pm 1, \ldots$

and $\left(-2, \frac{\pi}{6} + 2k\pi\right)$.

(c) (5 pts) Find the equation for the tangent line to the polar curve $r = \cos(3\theta) + \theta$ at $\theta = \pi$.

use $x = \cos\theta\,(\cos(3\theta) + \theta)$, and $\frac{dy}{dx} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\pi}$

$y = \sin\theta\,(\cos(3\theta) + \theta)$,

At $\theta = \pi$, $x = -1(-1+\pi) = 1 - \pi$, $y = 0$

$\left.\frac{dx}{d\theta}\right|_{\theta=\pi} = \left. -\sin\theta\,(\cos(3\theta)+\theta) + \cos\theta\,(-3\sin(3\theta) + 1)\right|_{\theta=\pi} = -1$

$\left.\frac{dy}{d\theta}\right|_{\theta=\pi} = \left. \cos\theta\,(\cos(3\theta)+\theta) + \sin\theta\,(-3\sin(3\theta)+1)\right|_{\theta=\pi} = 1 - \pi$

$\frac{dy}{dx} = (1-\pi)/-1 = \pi - 1$, tangent line is

$$y = (\pi - 1)(x + \pi - 1)$$

2) (a) (5 pts) For what values of k does $y(t) = e^{kt}$ solve $y'' - 3y' + 2y = 0$?

$\frac{dy}{dt} = ke^{kt}$, $\frac{d^2y}{dt^2} = k^2e^{kt}$, so

$y'' - 3y' + 2y = (k^2 - 3k + 2)e^{kt} = 0$

so since $e^{kt} \neq 0$, $k^2 - 3k + 2 = 0 = (k-2)(k-1)$

e^{kt} is a solution to the above differential equation when $k = 1, 2$.

(b) (7 pts) Use the method of separation of variables in order to solve the initial value problem $\frac{dy}{dt} = y^2 t$, $y(1) = 1$.

write the differential equation as $\frac{dy}{y^2} = t\,dt$, so $\int \frac{dy}{y^2} = \int t\,dt = \frac{t^2}{2} + c$

$\int \frac{dy}{y^2} = -y^{-1}$. Thus, $-y^{-1} = \frac{t^2}{2} + c$.

If $y(1) = 1$, then $-1 = \frac{1}{2} + c$, $c = -\frac{3}{2}$)

$-y^{-1} = \frac{t^2 - 3}{2}$, so $y = \frac{2}{3 - t^2}$.

3) (a) (5 pts) Determine whether the improper integral $\int_0^2 \frac{1}{x-1}\,dx$ is convergent or divergent. If it is convergent, calculate its value.

$$\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{dx}{x-1} + \int_1^2 \frac{dx}{x-1}.$$

$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln|1-x|\Big|_0^t = -\infty$$

Above integral does not exist since both improper integrals must exist.

(b) (7 pts) Use the integral test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is convergent or divergent. Be sure to check all hypotheses of the test.

Let $f(x) = \frac{1}{x\ln x}, x \geq 2$. Then f is decreasing on $[2,\infty)$ (since $x\ln x \geq 0$ and increasing on $[2,\infty)$). Also f is continuous and >0 on $[2,\infty)$ so by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ converges if and only if $\int_2^\infty \frac{1}{x\ln x}dx$ converges. Now $\int_2^\infty \frac{1}{x\ln x}$

$= \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x}$ = (use $u=\ln x$) $= \lim_{t\to\infty} \ln(\ln x)\Big|_2^t$

$= \infty$. Series Diverges by integral test

4) Calculate the following indefinite integrals.

(a) (5 pts) $\int x\cos x\,dx$. use integration by parts

Let $u = x$, $dv = \cos x\,dx$, so $du = dx$, $v = \sin x$

$\int u\,dv = \int x(\cos x)dx = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$

(b) (5 pts) $\int \frac{1}{x^3+x}\,dx$. $\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)}$.

(multiply by x^3+x to get)

$1 = A(x^2+1) + Bx^2 + Cx = (A+B)x^2 + Cx + A$

Equate coefficients, $A=1$, $C=0$, $A+B=0$, $B=-1$,

$\int \frac{dx}{x^3+x} = \int \frac{dx}{x} - \int \frac{x}{x^2+1}dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C$

(c) (5 pts) $\int \sin^3 x\cos^2 x\,dx$. Use $\sin^2 x + \cos^2 x = 1$ and let $u = \cos x$, $du = -\sin x\,dx$

$= -\int(1-u^2)u^2\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C$

$= -\frac{(\cos x)^3}{3} + \frac{(\cos x)^5}{5} + C$

5) Calculate the following definite integrals.

(a) (6 pts) $\displaystyle\int_0^1 \frac{x}{(9-x^2)^{3/2}}\,dx.$

Let $u = 9 - x^2$
$du = -2x\,dx$

$$\int \frac{x\,dx}{(9-x^2)^{3/2}} = -\frac{1}{2}\int u^{-3/2}\,du$$

$$= u^{-1/2} + c = (9-x^2)^{-1/2} + c$$

$$\text{so } \int_0^1 \frac{x\,dx}{(9-x^2)^{3/2}} = (9-x^2)^{-1/2}\Big|_0^1 = 8^{-1/2} - 9^{-1/2}$$

$$= \frac{1}{2\sqrt{2}} - \frac{1}{3}$$

(b) (6 pts) $\displaystyle\int_0^1 \frac{1}{(9-x^2)^{3/2}}\,dx.$

using

$x = 3\sin\theta$, $dx = 3\cos\theta$

$x = 0 = 3\sin\theta$ $(9-x^2)^{3/2} = 27(\cos\theta)^3$

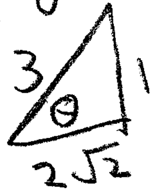
so $\theta = 0$

$x = 1 = 3\sin\theta$, $\theta = \arcsin\left(\frac{1}{3}\right)$

$$\int_0^1 \frac{dx}{(9-x^2)^{3/2}}$$

$$= \frac{1}{9}\int_0^{\arcsin(1/3)} \sec^2\theta\,d\theta = \frac{1}{9}\tan\theta\Big|_0^{\arcsin(1/3)}$$

$$= \frac{1}{9}\left(\frac{1}{2\sqrt{2}}\right) = \frac{1}{18\sqrt{2}}$$



6) (a) (5 pts) Find the length of the curve $y = 2x^{3/2}$ between $x = 1$ and $x = 2$.

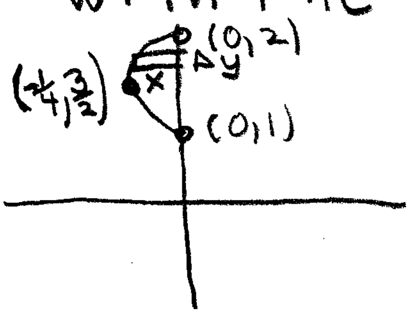
$$\frac{dy}{dx} = 3x^{\frac{1}{2}}, \quad \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+9x}$$

$$L = \int_1^2 \sqrt{1+9x}\,dx = \frac{2}{27}(1+9x)^{\frac{3}{2}}\Big)_1^2$$

$$= \frac{2}{27}\left[19^{\frac{3}{2}} - 10^{\frac{3}{2}}\right]$$

(b) (7 pts) Find the volume of the solid obtained by rotating the region bounded by the curves $x = 0$ and $x = y^2 - 3y + 2$ about the x-axis.

Complete the square to rewrite the above equation as $x = \left(y-\frac{3}{2}\right)^2 - \frac{1}{4}$ which is a parabola with vertex at $\left(-\frac{1}{4}, \frac{3}{2}\right)$. If $x=0$, $y-\frac{3}{2} = \pm\frac{1}{2}$ so $y = 2, 1$, and $(0,2)$, $(0,1)$ are the intersection of the parabola with the y axis. Using Shells,



$$\Delta V \approx 2\pi\,|x|\,y\,\Delta y$$

sum and take a limit to get

$$V = 2\pi\int_1^2 |x|\,y\,dy = -2\pi\int_1^2 (y^3 - 3y^2 + 2y)\,d$$

$$= -2\pi\left[\frac{y^4}{4} - y^3 + y^2\right]_1^2 = -2\pi\left[0 - \frac{1}{4}\right] = \frac{\pi}{2}$$

7) (a) (6 pts) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2n}$.

Let $a_n = \frac{(x-1)^n}{2n}$. Then $\frac{|a_{n+1}|}{|a_n|} = \frac{|x-1|^{n+1} 2n}{|x-1|^n 2(n+1)}$

$= \frac{|x-1|}{(1+\frac{1}{n})} \xrightarrow[n\to\infty]{} |x-1|$. So by the ratio test the above series diverges if $|x-1|>1$ and converges absolutely (so converges) if $|x-1|<1$. If $x=2$, series $= \sum_{n=1}^{\infty} \frac{1}{2n}$, diverge (harmonic series). If $x=0$, series converges by the alternating series test. Interval is $[0,2)$.

(b) (6 pts) Find the smallest number n such that the n-th degree Taylor polynomial $T_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}$ approximates e^x to within 0.05 for all x with $-1 \le x \le 1$. (Recall Taylor's inequality, which states that if $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$ and $R_n(x) = f(x) - T_n(x)$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ whenever $|x-a| \le d$.)

Note that if $f(x) = e^x$, then $f^{(n)}(x) = e^x$ and $|f^{(n)}(x)| = e^x \le e = f^{(n)}(1)$ on $[-1,1]$.

Want to find n so that $\frac{e \, 1^{(n+1)}}{(n+1)!} = \frac{e}{(n+1)!} \le .05$

or $20e \le (n+1)!$ Take

$n=4$.

8) In parts a and b, determine whether the given series converge or diverge. Be sure to cite an appropriate test to justify your answer.

(a) (4 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$.

$\dfrac{|\cos n|}{n^2+1} \le \dfrac{1}{n^2}$ for $n = 1, 2, \ldots$ So $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2+1}$ converges by comparison with the p series ($p=2$), $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$ converges absolutely so converges by the absolute convergent series test.

(b) (4 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2 2^n}$.

Let $a_n = \dfrac{n!}{n^2 2^n}$. Then $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!\, n^2 2^n}{n!\, (n+1)^2 2^{n+1}}$

$= (n+1) \dfrac{n^2}{(n+1)^2\, 2}$ since $((n+1)! = (n+1)\, n!)$

$= \dfrac{1(n+1)}{2(1+\frac{1}{n})^2} \to \infty$ as $n \to \infty$. Series diverges by the ratio test

(c) (4 pts) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sin\frac{1}{n+1} \right)$ converges or diverges. If it converges, find its sum.

This series telescopes:

$$\sum_{n=1}^{N} \sin\left(\tfrac{1}{n}\right) - \sin\left(\tfrac{1}{n+1}\right) = \sin 1 - \sin\frac{1}{N+1}$$

$\to \sin 1$ as $N \to \infty$. So the above series has sum $=$ to $\sin 1$.