MA 114 — Calculus II Spring 2011
Exam 1

This is a closed book exam. There are nine (9) problems on ten (10) pages (including this cover page). Check and be sure that you have a complete exam.

No books or notes may be used during the exam. You may use a graphing calculator provided that it does not have symbolic manipulation capabilities. In addition, any device capable of electronic communication (cell phone, pager, etc.) must be turned off and out of sight during the exam.

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. If you need more space then use the backs of the exam pages.

Show your work. Answers without justification will receive no credit. Partial credit for a problem will be given only when there is coherent written evidence that you have solved part of the problem. In particular, answers that are obtained simply as the output of calculator routines will receive no credit. Finally, be aware that it is not the responsibility of the grader to determine which part of your response is to be graded. Be sure to erase or mark out any work that you do not want graded.

Name: ____________________________________

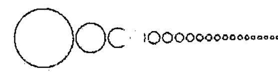
Section: ____________

Last four digits of student identification number: ____________

Problem	Score	Total
1		12
2		12
3		12
4		11
5		12
6		11
7		10
8		10
9		10
		100

1) (12 pts) In the diagram there are an infinite number of circles, $\{C_i\}_{i=1}^{\infty}$, stacked one atop the other . Circle C_1 is the largest, C_2 the next largest, etc. For each n, the radius of C_n is $\frac{1}{n^{\frac{3}{4}}}$ cm. 

(a) Write a formula for A_j, the area of the j^{th} circle (expressed in cm^2) . Show your work.



$$A_j = \frac{\pi}{j^{3/2}} \text{ cm}^2$$

(b) Write a formula for P_j, the circumference of the j^{th} circle (expressed in cm). Show your work.

$$P_j = \frac{2\pi}{j^{3/4}}$$

(c) Determine whether the total area of all of the circles is finite. That is, determine whether the series $\sum_{i=1}^{\infty} A_i$ is convergent or not. Explain your answer.

$A = \sum_{i=1}^{\infty} A_i = \pi \sum_{i=1}^{\infty} \frac{1}{i^{3/2}}$ converges since the series is a p series and $p > 1$ $(p = \frac{3}{2})$.

(d) Determine whether the total perimeter of all of the boxes is finite. That is, determine whether the series $\sum_{i=1}^{\infty} P_i$ is convergent or not. Explain your answer.

$P = \sum_{i=1}^{\infty} P_i = 2\pi \sum_{i=1}^{\infty} \frac{1}{i^{3/4}}$ diverges since the series is a p series and $p = \frac{3}{4} < 1$

2) (12 pts) Let $f(x) = \sum_{n=1}^{\infty} n^4 \frac{(2x)^n}{3^n}$.

(a) Calculate the interval of convergence of f. Be sure to check end points.

Let $a_n = \frac{n^4 (2x)^n}{3^n}$. Then $\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^4 |2x|^{n+1}}{3^{n+1}} \Big/ \left(\frac{n^4 |2x|^n}{3^n}\right)$

$= \frac{(n+1)^4 |2x|}{3n^4} = \left(1+\frac{1}{n}\right)^4 \left(\frac{2}{3}\right)|x| \xrightarrow[n\to\infty]{} \frac{2}{3}|x|$. By the ratio test the series for f converges absolutely so converges when $|x| < \frac{3}{2}$ and diverges when $|x| > \frac{3}{2}$. If $x = \pm\frac{3}{2}$ then $a_n = \pm n^4$ and $\lim_{n\to\infty} a_n \neq 0$ so the series diverges at $\pm\frac{3}{2}$ (by TFD). Radius of Convergence $= \frac{3}{2}$, Interval $= \left(-\frac{3}{2}, \frac{3}{2}\right)$.

(b) Calculate a power series representation for $f'(x)$ (i.e. express $f'(x)$ in the form $f'(x) = \sum_{n=1}^{\infty} a_n x^n$). Show your work.

$$f'(x) = \sum_{n=1}^{\infty} \frac{2n^5 (2x)^{n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{2^n n^5 x^{n-1}}{3^n}$$

(c) What is the radius of convergence of $f'(x)$? Explain your answer.

Series converges for $|x| < \frac{3}{2}$ since a Power Series and its derivative series (obtained by termwise differentiation) have the same radius of convergence.

3) (12 pts)
Complete the following.

(a) If $\sum_{n=1}^{\infty} a_n$ is a series and N is a positive integer then S_N, the N^{th} partial sum of S, is defined to be

$$\sum_{n=1}^{N} a_n = S_N$$

(b) If $\sum_{n=1}^{\infty} a_n$ is a series and L is a number then $\sum_{n=1}^{\infty} a_n = L$ means precisely

$$\lim_{N\to\infty} S_N = L$$

(c) The sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded below** means that there is a number M such that

$a_n \geq M$ for all n $(n = 1, 2, \ldots)$

(d) If $S = \sum_{n=1}^{\infty} c_n (x-a)^n$ is a power series then the **interval of convergence of S** is

the set of all points where the power series converges.

4) (11 pts)

Conside: the telescoping series $S = \sum_{k=1}^{\infty} \frac{1}{\ln(n+1)} - \frac{1}{\ln(n+2)}$.

(a) Expr ss the partial sum $S_N = \sum_{k=1}^{n} a_k$ in terms of N.

$$S_N = \frac{1}{\ln(2)} - \frac{1}{\ln(3)} + \frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(4)} - \frac{1}{\ln(5)} + \cdots + \frac{1}{\ln(N+1)} - \frac{1}{\ln(N+2)} = \frac{1}{\ln(2)} - \frac{1}{\ln(N+2)}$$

(b) Use our answer to part (a) and the definition of series convergence to show that the serie $\sum_{j=1}^{\infty} a_j$ converges.

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \left[\frac{1}{\ln 2} - \frac{1}{\ln(N+2)} \right] = \frac{1}{\ln 2}$$

(c) Calc late the exact value of $\sum_{j=1}^{\infty} a_j$. $= \frac{1}{\ln 2}$

5) (12 pts)

Given the Alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ or $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n > 0, n = 1, 2, \ldots$ If $a_{n+1} < a_n$ for all $n = 1, 2, \ldots$) and $\lim_{n\to\infty} a_n = 0$ then either of the above series converge.

(a) State the Alternating Series Test in full.

$\frac{1}{e^{n+1}} < \frac{1}{e^n}$ is equivalent to $e^n < e^{n+1}$ which is true since e^x is an increasing function.

$\lim_{n\to\infty} \frac{1}{e^n} = 0$ since e^n grows much faster than n and so $\frac{1}{e^n}$ goes to zero faster than $\frac{1}{n}$.

~~(b)~~ Carefully apply the alternating series test to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^n}$ to show that the series converges. You must verify all of the hypothesis.

Thus $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^n}$ converges by the AST.

(c) Apply the Alternating Series Estimation Theorem to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^n}$ and determine the least *integer* N such that the N^{th} partial sum S_N is within .01 of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{e^n}$

We need to find N so that "$\frac{1}{e^{N+1}} < \frac{1}{100}$" or $100 < e^{N+1}$. Take $N = 4$. then by the ASET: $\sum_{n=1}^{4} \frac{(-1)^{n+1}}{e^n}$ approximates the actual sum within $\frac{1}{100}$.

6) (11 pts) You must carefully explain your answers to each of the following.

(a) For which values of s is the series $\sum_{n=1}^{\infty} \frac{1}{n^{s^2-24}}$ divergent?

$s^2 - 24 \leq 1$ or $s^2 \leq 25$ so series diverges for $|s| \leq 5$ since then the above series will be a p series with $p \leq 1$,

(b) What are the values of s for which $\sum_{n=1}^{\infty} (2s+1)^n$ is convergent?

Converges for $|2s+1| < 1$ or $|s+\frac{1}{2}| < \frac{1}{2}$ or for

s in $(-1,0) = \{s \mid -1 < s < 0\}$

In fact for these valules the above series is a geometric series and has sum $\dfrac{2s+1}{1-(2s+1)} = -\dfrac{2s+1}{2s} = -1 - \dfrac{1}{2s}$.

For other values it is also a geometric series but $|2s+1| \geq 1$ so the series diverges.

7) (10 pts) Analyze each of the following series for convergence. Use one or mor convergence tests to justify your answer.
State the name of each convergence test that you use and explain carefully h w you apply it.

(a) Does $\displaystyle\sum_{n=1}^{\infty} \frac{n\sqrt{n}}{(n+1)(n+2)}$ converge or diverge? You must justify your ans ver.

Diverges by using the limit comparison test and comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$. In fact

$$\lim_{n\to\infty} \frac{\frac{n\sqrt{n}}{(n+1)(n+2)}}{\frac{1}{n^{\frac{1}{2}}}} = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})} = 1 \neq 0.$$ So both series diverge together.

(b) Does $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-2)^n}$ converge or diverge? You must justify your answer.

$a_n = \frac{n!}{(-2)^n}$ and $\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}}$

(since $(n+1)! = (n+1)\, n!$)

$$= \frac{(n+1)}{2} \xrightarrow[n\to\infty]{} \infty > 1.$$

Series diverges by the ratio test.

8) (10 pts Use one or more convergence tests to determine whether each of the following converg s or diverges. State the name of each convergence test that you use and explain carefull how you apply it.

(a) Doe $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4+3n+7}}{\sqrt{n}\,(n^3+2)}$ converge or diverge? You must justify your answer.

Series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{11}{n^{3/2}}$. In fact since $n < n^4$ for $n = 1, 2, 3, \ldots$ $\sqrt{n^4+3n+7} < \sqrt{n^4+3n^4+7n^4} < 11n^2$

and $\sqrt{n}(n^3+2) > \sqrt{n}\, n^3 = n^{7/2}$. Thus

$\dfrac{\sqrt{n^4+3n+7}}{\sqrt{n}(n^3-2)} < \dfrac{11n^2}{n^{7/2}} = \dfrac{11}{n^{3/2}}$. Series

$\displaystyle\sum_{n=1}^{\infty} \frac{11}{n^{3/2}} = 11 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges so the series in (a) converges by the comparison test.

(b) Doe $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{\sqrt{2n^2+10}}\right)^n$ converge or diverge? You must justify your answer.

Let $a_n = \left(\dfrac{n+1}{\sqrt{2n^2+10}}\right)^n$. Then

$a_n^{\frac{1}{n}} = \dfrac{n+1}{\sqrt{2n^2+10}} = \dfrac{n+1}{\sqrt{n^2}\sqrt{2+\frac{10}{n^2}}} = \dfrac{n+1}{n\sqrt{2+\frac{10}{n^2}}}$

$= \dfrac{(1+\frac{1}{n})}{\sqrt{2+\frac{10}{n^2}}} \xrightarrow[n\to\infty]{} \dfrac{1}{\sqrt{2}} < 1$ so the series in (b) converges by the root test.

9) (10 pts) Give examples of the following:. **You must justify your answers.**

(a) Consider the sequence of numbers $\{\frac{1}{n}\}_{n=1}^{\infty}$. Give

1. Give an upper bound for the above sequence.

upper bound $= 2$

2. Give a lower bound for the above sequence, and

lower bound $= 0$

3. Give a number which is neither an upper nor a lower bound for th above .

$\frac{1}{2}$

(b) A divergent series $\sum_{i=i}^{\infty} a_i$ for which $\sum_{i=i}^{\infty} a_i^2$ is convergent. ∞

Take $a_i = \frac{1}{i}$. Then $\sum_{i=1} \frac{1}{i}$ diverges but $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges.

(c) A series which fails to converge by the test for divergence.

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \cdots +$$