

This is a closed book exam. There are seven (7) problems on nine (9) pages (including this cover page). Check and be sure that you have a complete exam.

No books or notes may be used during the exam. You may use a graphing calculator provided that it does not have symbolic manipulation capabilities. In addition, *any* device capable of electronic communication (cell phone, pager, etc.) must be turned off and out of sight during the exam.

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. If you need more space then use the backs of the exam pages.

*Show your work.* Answers without justification will receive no credit. Partial credit for a problem will be given only when there is coherent written evidence that you have solved part of the problem. In particular, answers that are obtained simply as the output of calculator routines will receive no credit. Finally, be aware that it is not the responsibility of the grader to determine which part of your response is to be graded. Be sure to erase or mark out any work that you do not want graded.

Name: \_\_\_\_\_

Section: \_\_\_\_\_

Last four digits of student identification number: \_\_\_\_\_

Problem	Score	Total
1		15
2		10
3		15
4		15
5		15
6		15
7		15
		100

Some useful trigonometric identities are  $\sin 2\theta = 2 \sin \theta \cos \theta$ ,  $\cos^2 \theta = \frac{1+\cos(2\theta)}{2}$ ,  $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$ .  $\sin^2 \theta + \cos^2 \theta = 1$ ,  $1 + \tan^2 \theta = \sec^2 \theta$ ,  $1 + \cot^2 \theta = \csc^2 \theta$ .

- 1) (a) Find the Taylor series expansion of  $\sin x$  about  $x = \pi/2$ . Your answer should be in the form of an infinite series in which the general term is clearly indicated.

- (b) Given  $f(x) = (1 - x)^{-1/2}$ . Find the Maclaurin (or Taylor) polynomial,  $T_2(x)$ , corresponding to  $f$  and  $a = 0$ .

- (c) Let  $f, T_2$  be as in (b). Given that  $f'''(x) = \frac{15}{8}(1-x)^{-7/2}$  is increasing on  $[-1/2, 1/2]$ . Find the smallest number  $A$ , guaranteed by Taylor's inequality, for which  $|f(x) - T_2(x)| \leq A$  for all  $x$  in  $[-1/2, 1/2]$ . Indicate your reasoning.

- 2)** Find the area of the region bounded by the graph of  $y = \cos x$ , and the lines  $x = 0$ ,  $x = \pi$ ,  $y = 1/2$ . Show your work

- 3)** Given the region bounded by the graph of  $y = e^{x/2}$  and the lines  $x = 0$ ,  $x = 1$ ,  $y = 1$ .
- (a) Find the volume of the solid obtained from rotating this region about the  $x$  axis. Show your work.

- (b) Set up (do not evaluate) an integral(s) for the volume obtained from rotating this region about the  $y$  axis.

- 4) A swimming pool is 6 feet deep and has a square base which is 50 feet on each side. Given that the density of water is  $62.5 \text{ lbs/ft}^3$ .
- (a) If the pool is completely full how much work is required to empty the pool by pumping water over the sides.
- (b) If the pool is half full how much work is required to empty the pool by pumping water over the sides.

- 5) Find the following integrals. Indicate clearly which integration method you are using. For example if integration by parts is used, write: let  $u = \dots, dv = \dots$

(a)  $\int x e^{3x} dx$

(b)  $\int_0^{\sqrt{\pi}} x \sin(x^2) dx$

6) Find the following integrals. Indicate clearly which integration method you are using.

(a)  $\int x^3 \sqrt{4 - x^2} dx$

(b)  $\int_0^{2\pi} \cos^2(3x) dx$



7) Find the following integrals. Indicate clearly which integration technique you are using.

(a)  $\int \frac{\sqrt{x^2 - 9}}{x} dx$

(b)  $\int_0^1 \arctan x \, dx$

1. (a) There are two ways to do this problem. One way is to note that the Macclaurin expansion of  $\cos w$  is  $\sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!}$  and this series converges for  $-\infty < w < \infty$ . Then use  $w = \pi/2 - x$  and

$\sin x = \cos(\pi/2 - x)$  to get,  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x - \pi/2)^{2n}}{(2n)!}$ . The second way is to calculate this expansion directly. Indeed if  $f(x) = \sin x$  then one calculates  $f^{(2n)}(\pi/2) = (-1)^n \sin x|_{x=\pi/2} = (-1)^n$  while  $f^{(2n+1)}(\pi/2) = (-1)^n \cos x|_{x=\pi/2} = 0$ . Using these numbers in the expression for a Taylor series about  $\pi/2$  one gets the above series.

(b) If  $f(x) = (1 - x)^{-1/2}$ , then  $f'(x) = (1/2)(1 - x)^{-3/2}$  and  $f''(x) = (1/2)(3/2)(1 - x)^{-5/2}$ . Thus  $f(0) = 1$ ,  $f'(0) = 1/2$ , and  $f''(0) = 3/4$ . Then  $T_2(x) = 1 + x/2 + 3x^2/8$ .

(c) Note that  $f'''(x)$  assumes its maximum on  $[-1/2, 1/2]$  at  $x = 1/2$  so  $|f'''(x)| \leq \frac{15}{8}2^{7/2} = 15\sqrt{2}$  on  $[-1/2, 1/2]$ . Taylor's inequality gives  $|f(x) - T_2(x)| \leq 15\sqrt{2}(|x|^3/6) \leq 5\sqrt{2}/16 = A \approx 0.44194$ .

2. Note that  $\cos x \geq 1/2$  for  $0 \leq x \leq \pi/3$  while  $\cos x < 1/2$  for  $\pi/3 < x < \pi$ . Thus if  $A$  denotes the area of the region bounded by the graph of  $y = \cos x$ , and the

lines  $x = 0$ ,  $x = \pi$ ,  $y = 1/2$ , then  $A = \int_0^{\pi/3} (\cos x - 1/2) dx + \int_{\pi/3}^{\pi} (1/2 - \cos x) dx = \sin x - x/2|_0^{\pi/3} - (\sin x - x/2)|_{\pi/3}^{\pi} = \sqrt{3} + \pi/6 \approx 2.2556$ .

3. (a) Using washers,  $V = \pi \int_0^1 (e^x - 1) dx = \pi(e^x - x)|_0^1 = \pi(e - 2) \approx 2.2565$

(b) By shells,  $V = 2\pi \int_0^1 x(e^{x/2} - 1) dx$ .

By washers,  $V = \pi \int_1^{e^{1/2}} [1 - 4(\ln y)^2] dy$ .

4. Let  $x$  be the distance from the top of the pool. The volume of water in the pool which is between  $x$  and  $x + \Delta x$  from the top is  $2500\Delta x$  and its weight is  $(2500)(62.5) \Delta x$ . The work required to lift this weight a height  $x$  is  $x(2500)(62.5)\Delta x$ . Summing over all such slabs and

taking a limit gives  $W = \int_0^6 x(2500)(62.5) dx = (1250)(62.5)x^2|_0^6 = 2,812,500$  ft/lbs.

(b) The water in the pool now lies between height  $x = 3$  and  $x = 6$ . Arguing as in (a) one gets  $W = \int_3^6 (2500)(62.5)x dx = (1250)(62.5)x^2|_3^6 = 2,109,375$  ft/lbs.

5. (a) Use integration by parts,  $u = x$ ,  $dv = e^{3x}$ . So  $du = dx$  and  $v = \int e^{3x} dx = \frac{e^{3x}}{3}$ . Thus  $\int x e^{3x} dx = (1/3)x e^{3x} - (1/3) \int e^{3x} dx = (1/3)x e^{3x} - (1/9)e^{3x} + c$ .

(b) Use substitution,  $u = x^2$ ,  $du = 2x dx$ . So  $u = 0$  when  $x = 0$  and  $u = \pi$  when  $x = \sqrt{\pi}$ . Changing variables gives,  $\int_0^{\sqrt{\pi}} x \sin(x^2) dx = (1/2) \int_0^{\pi} \sin(u) du = -\frac{1}{2} \cos u|_0^{\pi} = 1$ .

6. (a) let  $x = 2 \sin \theta$ . Then  $\sqrt{4 - x^2} = \sqrt{4(1 - \sin^2 \theta)} = 2|\cos \theta|$ . Suppose  $\cos \theta > 0$ . Then  $dx = 2 \cos \theta d\theta$  and by the above substitution,

$$\int x^3 \sqrt{4 - x^2} dx = 32 \int \sin^3 \theta \cos^2 \theta d\theta = 32 \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta.$$

To do the last integral let  $u = \cos \theta$ , so  $du = -\sin \theta d\theta$  and the integral becomes,

$$32 \int (u^2 - 1)u^2 du = 32(u^5/5 - u^3/3) + c = 32[(\cos \theta)^5/5 - (\cos \theta)^3/3] + c.$$

Now from the definition of  $\sin \theta$  one sees that  $\theta$  is an angle in a right triangle with hypotenuse 2, opposite side  $x$ , and adjacent side,  $\sqrt{4 - x^2}$ . Thus  $\cos \theta = (1/2)\sqrt{4 - x^2}$ . Putting this in the above expression gives,  $\int x^3 \sqrt{4 - x^2} dx = \frac{1}{5}(4 - x^2)^{5/2} - \frac{4}{3}(4 - x^2)^{3/2} + c$ .

If  $\cos \theta < 0$  one gets the same answer by carefully going through the argument and using  $\cos \theta = -\sqrt{4 - x^2}$  to convert back to an expression in  $x$ .

(b) Use  $\cos^2(3x) = \frac{1 + \cos(6x)}{2}$ . Then

$$\int_0^{2\pi} \cos^2(3x) dx = (1/2) \int_0^{2\pi} (1 + \cos(6x)) dx = \frac{x}{2} + \frac{\sin(6x)}{12} \Big|_0^{2\pi} = \pi.$$

7. (a) Let  $x = 3 \sec \theta$ . Then  $\sqrt{x^2 - 9} = \sqrt{9(\sec^2 \theta - 1)} = 3|\tan \theta|$ . Also  $dx = 3 \sec \theta \tan \theta d\theta$ .

$$\text{Assuming } \tan \theta \geq 0 \text{ gives } \int \frac{\sqrt{x^2 - 9}}{x} dx = 3 \int \tan^2 \theta d\theta$$

$$= 3 \int (\sec^2 \theta - 1) d\theta = 3 \tan \theta - 3\theta + c.$$

Note from the definition of  $\sec \theta$  that  $\theta$  is an angle in a right triangle whose hypotenuse is  $x$  adjacent side is 3 and opposite side is  $\sqrt{x^2 - 9}$ . Thus  $\tan \theta = \frac{1}{3}\sqrt{x^2 - 9}$ . Using this in the

$$\text{above finally gives, } \int \frac{\sqrt{x^2 - 9}}{x} dx = \sqrt{x^2 - 9} - 3 \operatorname{arcsec}(x/3) + c.$$

(b) Use integration by parts,  $u = \arctan x, dv = dx$ . Then  $du = \frac{dx}{1+x^2}, v = x$ . Integrating by

$$\text{parts gives, } \int_0^1 \arctan x dx = x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx = \pi/4 - (1/2) \ln(1+x^2) \Big|_0^1$$

$$= \pi/4 - (1/2) \ln(2) \approx .4388.$$

The last integral was done using the substitution,  $u = 1 + x^2, du = 2x dx$ .