Erratum: Computing Singular Values of Diagonally Dominant Matrices to High Relative Accuracy

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Abstract

We correct an error in the proof of Theorem 3 in [1] for the case of the column diagonal dominance pivoting.

Theorem 3 in [1] provides the forward error bound for the *LDU* factorization by Algorithm 1 with either the diagonal pivoting strategy or the column diagonal dominance pivoting strategy. There is an error in the proof for the case of the column diagonal dominance pivoting. Specifically, in line 5 of page 2225, we have used that

$$\frac{|\delta_{ik}^{(k)}|}{a_{kk}^{(k)}} \le \phi(k)\epsilon_1.$$

This is true if the diagonal pivoting is used as we have $\hat{a}_{ii}^{(k)} \leq \hat{a}_{kk}^{(k)}$ and hence

$$\frac{|\delta_{ik}^{(k)}|}{a_{kk}^{(k)}} \le \frac{\phi(k)\epsilon_1 a_{ii}^{(k)}}{a_{kk}^{(k)}} \le \phi(k)\epsilon_1 \frac{\widehat{a}_{ii}^{(k)}(1-\xi(k)\epsilon_1)^{-1}}{\widehat{a}_{kk}^{(k)}(1+\xi(k)\epsilon_1)^{-1}} \le \phi(k)\epsilon_1.$$

If the column diagonal dominance pivoting is used, $\hat{a}_{ii}^{(k)} \leq \hat{a}_{kk}^{(k)}$ may not be true. So, Theorem 3 as stated is proved only under the assumption that the diagonal pivoting is used. When the column diagonal dominance pivoting is used, we will show that the theorem is still true if the ∞ -norm in the bound for L is replaced by the 1-norm. Here we present some additional arguments to prove this case.

Consider the column diagonal dominance pivoting strategy where, at the k-th step of the Gaussian elimination, we simultaneously permute the rows and columns so that either the pivot entry

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is maximal among those with diagonally dominant columns or the pivot column has maximal diagonal dominance. In carrying out the error analysis, we assume that the matrix A has been permuted in advance by the accumulated permutation matrix P as determined by implementing Algorithm 1 (with the column diagonal dominance pivoting) in a floating point arithmetic. Recall that $\widehat{A}^{(k)} = (\widehat{a}_{ij}^{(k)})$ denote the computed matrix after the (k-1)-st Gaussian elimination and $\widehat{v}^{(k)} = (\widehat{v}_i^{(k)})$ denote the computed matrix after the (k-1)-st Gaussian elimination and $\widehat{v}^{(k)} = (\widehat{v}_i^{(k)})$ denote the computed diagonally dominant part. Let $\widehat{w}_i^{(k)} = \left(\widehat{v}_i^{(k)} + \sum_{j=k,j\neq i}^n |\widehat{a}_{ij}^{(k)}|\right) - \sum_{j=k,j\neq i}^n |\widehat{a}_{ji}^{(k)}|$ be the diagonal dominant part for the *i*-th column. Then $\widehat{A}^{(k)}$ is already arranged with the column diagonal dominance pivoting, i.e.

$$\widehat{a}_{kk}^{(k)} = \max\{\widehat{a}_{ii}^{(k)} : \widehat{w}_i^{(k)} \ge 0, i \ge k\},\tag{1}$$

or

$$\widehat{w}_{k}^{(k)} = \max\{\widehat{w}_{i}^{(k)} : \widehat{w}_{i}^{(k)} \ge 0, i \ge k\},\tag{2}$$

where we have assumed that the sign of $\hat{w}_i^{(k)}$ is correctly computed for determining the column for pivoting.

Recall that N is the maximal integer such that $N \leq n-1$ and $a_{\ell\ell}^{(\ell)} > 0$ for $1 \leq \ell \leq N$. We first have the following result.

Lemma 1 For $1 \le k \le N$, if $\widehat{A}^{(k)}$ is arranged so that (1) (or (2)) holds, then we have

$$\sum_{i=k+1}^{n} (|\hat{a}_{ik}^{(k)}| + \hat{v}_{i}^{(k)}) \le n(1+\epsilon_2)\hat{a}_{kk}^{(k)}$$

Proof Let $\tilde{a}_{ii}^{(k)} := \hat{v}_i^{(k)} + \sum_{j=k, j \neq i}^n |\hat{a}_{ij}^{(k)}|$. Then $\hat{a}_{kk}^{(k)} = fl\left(\hat{v}_i^{(k)} + \sum_{j=k, j \neq i}^n |\hat{a}_{ij}^{(k)}|\right) = \tilde{a}_{ii}^{(k)}(1 + \epsilon_2)$. It follows from $|\hat{a}_{ik}^{(k)}| + \hat{v}_i^{(k)} = \tilde{a}_{ii}^{(k)} - \sum_{i=k+1, j \neq i}^n |\hat{a}_{ij}^{(k)}|$

that

$$\begin{split} \sum_{i=k+1}^{n} \left(|\widehat{a}_{ik}^{(k)}| + \widehat{v}_{i}^{(k)} \right) &= \sum_{i=k+1}^{n} \widetilde{a}_{ii}^{(k)} - \sum_{i=k+1}^{n} \sum_{j=k+1, j \neq i}^{n} |\widehat{a}_{ij}^{(k)}| \\ &= \sum_{i=k+1}^{n} \sum_{j=k, j \neq i}^{n} |\widehat{a}_{ji}^{(k)}| + \sum_{i=k+1}^{n} \widehat{w}_{i}^{(k)} - \sum_{i=k+1}^{n} \sum_{j=k+1, j \neq i}^{n} |\widehat{a}_{ij}^{(k)}| \\ &= \sum_{i=k+1}^{n} |\widehat{a}_{ki}^{(k)}| + \sum_{i=k+1}^{n} \widehat{w}_{i}^{(k)} \\ &\leq \widetilde{a}_{kk}^{(k)} + \sum_{\widehat{w}_{i}^{(k)} \geq 0} \widehat{w}_{i}^{(k)}. \end{split}$$

If using column diagonal dominance pivoting (1), we have

$$\sum_{i=k+1}^{n} \left(|\widehat{a}_{ik}^{(k)}| + \widehat{v}_{i}^{(k)} \right) \leq \widetilde{a}_{kk}^{(k)} + \sum_{\widehat{w}_{i}^{(k)} \ge 0} \widetilde{a}_{ii}^{(k)} = \widehat{a}_{kk}^{(k)} (1 + \epsilon_{2}) + \sum_{\widehat{w}_{i}^{(k)} \ge 0} \widehat{a}_{ii}^{(k)} (1 + \epsilon_{2})$$
$$\leq n(1 + \epsilon_{2}) \widehat{a}_{kk}^{(k)}.$$

If using column diagonal dominance pivoting (2), we have

$$\sum_{i=k+1}^{n} \left(|\widehat{a}_{ik}^{(k)}| + \widehat{v}_{i}^{(k)} \right) \leq \widetilde{a}_{kk}^{(k)} + \sum_{\widehat{w}_{i}^{(k)} \ge 0} \widehat{w}_{k}^{(k)} \le n \widetilde{a}_{kk}^{(k)}$$
$$= n(1+\epsilon_{2})\widehat{a}_{kk}^{(k)}.$$

We now present the corrected version of Theorem 4 in [1] for the case of column diagonal dominance pivoting, which replaces the ∞ -norm in the bound for L by the 1-norm.

Theorem 1 Let $\hat{L} = [\hat{l}_{ik}]$, $\hat{D} = diag\{\hat{d}_i\}$ and $\hat{U} = [\hat{u}_{ik}]$ be the computed factors of LDUfactorization of $\mathcal{D}(A_D, v)$ by Algorithm 1 and $L = [l_{ik}]$, $D = diag\{d_i\}$ and $U = [u_{ik}]$ are the corresponding factors computed exactly. Assume that the column diagonal dominance pivoting is used so that (1) (or (2)) holds. We have

$$\begin{aligned} \|\widehat{L} - L\|_{1} &\leq \left(n\nu_{n-1}\mathbf{u} + \mathcal{O}(\mathbf{u}^{2})\right)\|L\|_{1}, \\ |\widehat{d}_{i} - d_{i}| &\leq \left(\xi_{n-1}\mathbf{u} + \mathcal{O}(\mathbf{u}^{2})\right)d_{i}, \text{ for } 1 \leq i \leq n, \\ |\widehat{U} - U\|_{\infty} &\leq \left(\nu_{n-1}\mathbf{u} + \mathcal{O}(\mathbf{u}^{2})\right)\|U\|_{\infty}, \end{aligned}$$
(3)

where $\nu_{n-1} \leq 6 \cdot 8^{n-1} - 2$ and $\xi_{n-1} \leq 5 \cdot 8^{n-1} - \frac{5}{2}$.

Proof We only prove the bound (3) for L. The proof for other parts is contained in the original proof [1, p.2225]. As in the proof of Theorem 3 in [1, p.2225], using Lemma 5 of [1] and the notations there, we have for $i \ge k + 1$, $|\hat{a}_{ik}^{(k)}| \le \hat{a}_{kk}^{(k)}$ and

$$\begin{aligned} |\hat{l}_{ik} - l_{ik}| &\leq \left| \frac{\hat{a}_{ik}^{(k)}}{\hat{a}_{kk}^{(k)}} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} \right| + \frac{|\hat{a}_{ik}^{(k)}|}{\hat{a}_{kk}^{(k)}} \mathbf{u} \\ &\leq \left| \frac{\delta_{ik}^{(k)}|}{a_{kk}^{(k)}} + \frac{|\hat{a}_{ik}^{(k)}\delta_{kk}^{(k)}|}{\hat{a}_{kk}^{(k)}a_{kk}^{(k)}} + \frac{|\hat{a}_{ik}^{(k)}|}{\hat{a}_{kk}^{(k)}} \mathbf{u} \\ &\leq \frac{\phi(k)\epsilon_1(\hat{v}_i^{(k)} + |\hat{a}_{ik}^{(k)}|)}{a_{kk}^{(k)}} + \xi(k)\epsilon_1 + \epsilon_1 \end{aligned}$$

where we have used $|\delta_{ik}^{(k)}| \leq \phi(k)\epsilon_1(\hat{v}_i^{(k)} + |\hat{a}_{ik}^{(k)}|)$ which follows from Lemma 5 of [1] as

$$\begin{aligned} |\delta_{ik}^{(k)}| &\leq \phi(k)\epsilon_1(v_i^{(k)} + |a_{ik}^{(k)}|) \\ &\leq \phi(k)\epsilon_1(\widehat{v}_i^{(k)} + |\widehat{a}_{ik}^{(k)}|) + \phi(k)\epsilon_1(|\delta_i^{(k)}| + |\delta_{ik}^{(k)}|) \\ &\leq \phi(k)\epsilon_1(\widehat{v}_i^{(k)} + |\widehat{a}_{ik}^{(k)}|) + \phi(k)\epsilon_1(\psi(k)\epsilon_1\widehat{v}_i^{(k)} + |\delta_{ik}^{(k)}|). \end{aligned}$$

Now, by Lemma 1, we have $\sum_{i=k+1}^{n} (|\hat{a}_{ik}^{(k)}| + \hat{v}_i^{(k)}) \le n(1+\epsilon_2)\hat{a}_{kk}^{(k)}$. Then

$$\begin{split} \sum_{i=k+1}^{n} |\hat{l}_{ik} - l_{ik}| &\leq \phi(k)\epsilon_1 \frac{\sum_{i=k+1}^{n} (\hat{v}_i^{(k)} + |\hat{a}_{ik}^{(k)}|)}{a_{kk}^{(k)}} + \sum_{i=k+1}^{n} \xi(k)\epsilon_1 + \sum_{i=k+1}^{n} \epsilon_1 \\ &\leq \phi(k)\epsilon_1 \frac{n(1+\epsilon_2)|\hat{a}_{kk}^{(k)}|}{a_{kk}^{(k)}} + (n-1)\xi(k)\epsilon_1 + (n-1)\epsilon_1 \\ &= n\phi(k)\epsilon_1 + (n-1)\xi(k)\epsilon_1 + (n-1)\epsilon_1 \\ &\leq n\nu_{n-1}\epsilon_1 \end{split}$$

where we have used $\widehat{a}_{kk}^{(k)} \leq (1 + \xi(k)\epsilon_1)a_{ii}^{(k)}$ and $\nu_{n-1} = 2\phi(n-1) + \psi(n-1) + 3$. Since $||L||_1 \geq 1$, we have thus $||\widehat{L} - L||_1 \leq n\nu_{n-1}\epsilon_1 ||L||_1$ and the theorem is proved.

References

Q. YE, Computing Singular Values of Diagonally Dominant Matrices to High Relative Accuracy, Math. Comp. 77(2008), 2195-2230.