RELATIVE PERTURBATION BOUNDS FOR EIGENVALUES OF SYMMETRIC POSITIVE DEFINITE DIAGONALLY DOMINANT MATRICES

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Abstract. For a symmetric positive semi-definite diagonally dominant matrix, if its off-diagonal entries and its diagonally dominant parts for all rows (which are defined for a row as the diagonal entry subtracted by the sum of absolute values of off-diagonal entries in that row) are known to a certain relative accuracy, we show that its eigenvalues are known to the same relative accuracy. Specifically, we prove that if such a matrix is perturbed in a way that each off-diagonal entry and each diagonally dominant part have relative errors bounded by some ϵ, then all its eigenvalues have relative errors bounded by ϵ. The result is extended to the generalized eigenvalue problem.

Keywords: relative perturbation, eigenvalues, diagonal dominant matrix,
AMS Subject Classifications: 65F35, 15A42,

1. Introduction. The study of relative perturbation theory and high relative accuracy algorithms has been a subject of great interest for many years, see [7, 12, 13] for an overview. For the matrix eigenvalue or singular value problems, by restricting perturbations to those that preserve certain structure and are small entrywise, the perturbation bounds could be strengthened and thus even some small singular values or eigenvalues can be guaranteed to have small relative perturbations, see [5, 6, 7, 9, 10, 14, 16, 15, 19] for some of the references. We note that such results can only be established by considering matrices perturbed within certain classes and in some cases, the matrices may need to be re-parameterized, see [7, 9, 14] for example.

In this paper, we develop a relative perturbation theory for eigenvalues of symmetric positive semi-definite diagonally dominant matrices (or symmetric diagonally dominant matrices with nonnegative diagonals). Diagonally dominant matrices arise in a large variety of applications and form one of the most well-studied class of matrices, see [18] for some recent interests. While the property of diagonal dominance has traditionally been used more in solving linear systems, in recent years, this is also exploited for eigenvalue computations. In [3], Barlow and Demmel developed entrywise perturbation analysis and algorithms for the eigenvalues of symmetric scaled diagonally dominant matrices. Their perturbation results [3] show that the relative perturbations on eigenvalues, when each entry of the matrix has small relative perturbation, depend on a condition number, which is essentially related to the diagonal dominance. In [1, 2], Alfa, Xue and Ye showed that the smallest eigenvalue of a diagonally dominant M-matrix is determined and can be computed to high relative accuracy without any condition number, if the row sums (i.e., the diagonally dominant parts) are known to high relative accuracy. Under the same assumptions, Demmel and Koev [8] showed that all singular values of a diagonally dominant M-matrix are determined and can be computed to high relative accuracy. More refined results on diagonally dominant M-matrices are given by Peña [17]. Other related perturbation results include those for M-matrices by Elsner [11], Xue [21], and Xue and Jiang [20], which all contain some condition numbers. Note that an M-matrix can be scaled to become a diagonally dominant M-matrix.

Here, we shall prove that if a symmetric positive semi-definite diagonally dominant matrix $A = [a_{ij}]$ is perturbed symmetrically with each off-diagonal entry a_{ij} and each

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diagonally dominant part ($v_i := a_{ii} - \sum_{j\neq i}|a_{ij}|$) having relative error bounded by ϵ, then the relative error of each eigenvalue is bounded exactly by ϵ. We shall also extend our results to the generalized eigenvalue problem. Compared with the results of [3], our perturbation bound is independent of any condition number. Compared with that of [2, 8], we do not require the matrix to be an M-matrix but rather we require symmetry. Also our bound is sharp and is valid for all eigenvalues.

We remark that the key to obtain the strong bound is to consider the diagonal dominant parts, replacing the diagonal entries, as the parameters representing such matrices. Namely, the eigenvalues may not be determined to high relative accuracy by the entries of A, but they are so determined by its off-diagonal entries and the diagonal dominant parts. This parametrization is originally introduced in Alfa, Xue and Ye [1, 2] for diagonally dominant M-matrices. We concentrate on the perturbation theory in this paper but consider algorithms that compute all eigenvalues to the order of machine precision in a separate work [22].

The rest of this paper is organized as follows. We first give in Section 2 some definitions and preliminary results. We then present the perturbation results in Section 3.

2. Preliminaries and notation. Throughout this paper, we shall use the following notations. Given a matrix $A = [a_{ij}]$, we use $|A| = [|a_{ij}|]$ and $\text{sign}(A) = [\text{sign}(a_{ij})]$ where $\text{sign}(x)$ denotes the sign of x with $\text{sign}(0) = 1$. Given a vector $v = [v_i]$, $\text{diag}\{v\}$ is the diagonal matrix with the entries of v on its diagonal. $A \geq 0$ denotes that A is symmetric positive semi-definite and $A \geq B$ denotes that $A - B$ is symmetric positive semi-definite.

The basis of our relative perturbation theory is a re-parametrization of the matrices by its off-diagonal entries and its diagonal dominant parts. This is originally introduced in [1, 2] for diagonal dominant M-matrices and can be done for a general matrix as follows.

DEFINITION 2.1. *Given an $n \times n$ matrix $M = [m_{ij}]$ and an n-vector $v = [v_i]$, we use $\mathcal{D}(M, v)$ to denote the matrix $A = [a_{ij}]$ whose off-diagonal entries are the same as M and whose ith diagonal entry is $a_{ii} = v_i + \sum_{j\neq i}|m_{ij}|$. Namely, we write*

$$A = \mathcal{D}(M, v), \tag{2.1}$$

and call it the representation of A by diagonally dominant parts v, if

$$a_{ij} = m_{ij} \text{ for } i \neq j; \text{ and } a_{ii} = v_i + \sum_{j\neq i}|m_{ij}|.$$

Note that the diagonal entries of M, if given, are not used in defining the matrix $\mathcal{D}(M, v)$. Now, given a matrix $A = [a_{ij}]$, we denote by A_D the matrix whose off-diagonal entries are the same as A and whose diagonal entries are zero. Then, letting $v_i = a_{ii} - \sum_{j\neq i}|a_{ij}|$ and $v = (v_1, v_2, \cdots, v_n)^T$, we have

$$A = \mathcal{D}(A_D, v)$$

as the representation of A by diagonally dominant parts. In this way, the parameters defining A are those of A_D (i.e., the off-diagonal entries of A) and v (i.e., the diagonally dominant parts). The diagonal entries are not used to define A in this representation.

DEFINITION 2.2. *A matrix $A = [a_{ij}]$ is said to be diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i}|a_{ij}|$ for all i.*

The diagonally dominant matrix defined here is sometimes referred to as being weakly diagonally dominant. A matrix represented in $A = \mathcal{D}(A_D, v)$ is diagonally dominant with nonnegative diagonals if and only if $v_i \geq 0$ for all i.

Throughout we shall consider symmetric matrices $\mathcal{D}(A_D, v)$ with $v_i \geq 0$ for all i, i.e., symmetric diagonally dominant matrices with nonnegative diagonals. Clearly, a matrix is symmetric diagonally dominant with nonnegative diagonals if and only if it is symmetric positive semi-definite and diagonally dominant.

3. Relative Perturbation Bounds. We present relative perturbation bounds for symmetric positive semi-definite diagonally dominant matrices, which are represented in the form $A = \mathcal{D}(A_D, v)$ with $v_i \geq 0$ for all i.

We first introduce some notations. For $A = [a_{ij}]$, let

$$T_i = \mathrm{diag}(a_{1,i+1}, a_{2,i+2}, \cdots, a_{n-i,n}), \tag{3.1}$$

$$D_i = \begin{array}{c} \begin{matrix} \scriptstyle n-i & \scriptstyle i \end{matrix} \\ \begin{pmatrix} |T_i| & 0 \\ 0 & 0 \end{pmatrix} \end{array} \begin{matrix} \scriptstyle n-i \\ \scriptstyle i \end{matrix}, \quad N_i = \begin{array}{c} \begin{matrix} \scriptstyle n-i & \scriptstyle i \end{matrix} \\ \begin{pmatrix} 0 & 0 \\ T_i & 0 \end{pmatrix} \end{array} \begin{matrix} \scriptstyle i \\ \scriptstyle n-i \end{matrix}, \tag{3.2}$$

and

$$M_i = \mathrm{sign}(N_i), \quad L_i = I + M_i. \tag{3.3}$$

In words, T_i is the diagonal matrix whose diagonal is given by the ith diagonal of A above the main diagonal; D_i is the $n \times n$ diagonal matrix whose diagonals are absolute values of T_i, extended by zeros; and N_i is the matrix obtained from A by striking out all entries except the ith diagonal below the main diagonal. Clearly, L_i is lower triangular with ± 1 on the $i-$th diagonal.

We first give a lemma that gives a LDL^T factorization of a symmetric banded matrix that has a single band and zero diagonal dominant part. A feature of this factorization is that the parameters defining the matrix is entirely contained in the D matrix.

LEMMA 3.1. *Let A_i be the symmetric matrix whose entries are zero everywhere except in the ith diagonal, which is $a_{1,i+1}, a_{2,i+2}, \cdots, a_{n-i,n}$, and in the main diagonal, whose jth entry is the sum of absolute values of the off-diagonal entries in the jth row, i.e., $A_i = \mathcal{D}(N_i + N_i^T, 0)$ in the representation (2.1). Then, we have $A_i = L_i D_i L_i^T$.*

Proof. Since the diagonally dominant part is 0, we can write

$$A_i = N_i + N_i^T + \begin{pmatrix} 0 & 0 \\ 0 & |T_i| \end{pmatrix} + \begin{pmatrix} |T_i| & 0 \\ 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} L_i D_i L_i^T &= (I + M_i) D_i (I + M_i^T) \\ &= D_i + M_i D_i + D_i M_i^T + M_i D_i M_i^T \\ &= D_i + N_i + N_i^T + N_i M_i^T = A_i. \end{aligned}$$

where we note that

$$N_i M_i^T = \begin{pmatrix} 0 & 0 \\ 0 & |T_i| \end{pmatrix}.$$

□

There is a similar result for the case $i = 1$ presented in [4, Example 5.1,p.196], which was the inspiration of this work. We next decompose A into a sum of banded matrices with a single band and hence, using the above lemma, decompose it into a sum of LDL^T factorizations.

LEMMA 3.2. *Let $A = [a_{ij}]$ be a symmetric matrix and let $A = \mathcal{D}(A_D, v)$ be its representation in (2.1) with $v = [v_1, v_2, \cdots, v_n]^T$. Then*

$$A = V_0 + L_1 D_1 L_1^T + \cdots + L_{n-1} D_{n-1} L_{n-1}^T \tag{3.4}$$

where $V_0 = diag\{v_1, \cdots, v_n\}$, and L_i, D_i (for $1 \le i \le n-1$) are as defined in (3.3) and (3.2).

Proof. Let $A_i = \mathcal{D}(N_i + N_i^T, 0)$ as defined in Lemma 3.1. Then the off-diagonal entries of $\Sigma_{i=1}^{n-1} A_i$ are the same as those of A. Since each A_i has zero diagonal dominant part, it is easy to see that $\Sigma_{i=1}^{n-1} A_i$ has also zero diagonal dominant part. Thus, we have

$$\sum_{i=1}^{n-1} A_i = \mathcal{D}(A, 0).$$

Now, with O denoting the zero matrix, we have

$$A = \mathcal{D}(O, v) + \mathcal{D}(A, 0) = V_0 + \Sigma_{i=1}^{n-1} A_i$$

and hence (3.4) follows from Lemma 3.1. □

To clearly see the decomposition (3.4), we show an example of 3×3 matrix $A = \mathcal{D}(A_D, v)$ as the following

$$\begin{aligned}
A &= \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\
&= \begin{pmatrix} v_1 & & \\ & v_2 & \\ & & v_3 \end{pmatrix} + \begin{pmatrix} |a_{21}| + |a_{31}| & a_{21} & a_{31} \\ a_{21} & |a_{21}| + |a_{32}| & a_{32} \\ a_{31} & a_{32} & |a_{31}| + |a_{32}| \end{pmatrix} \\
&= \begin{pmatrix} v_1 & & \\ & v_2 & \\ & & v_3 \end{pmatrix} + \begin{pmatrix} 1 & & \\ s_{21} & 1 & \\ & s_{32} & 1 \end{pmatrix} \begin{pmatrix} |a_{21}| & & \\ & |a_{32}| & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & s_{21} & \\ & 1 & s_{32} \\ & & 1 \end{pmatrix} \\
&\quad + \begin{pmatrix} 1 & & \\ 0 & 1 & \\ s_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} |a_{31}| & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & s_{31} \\ & 1 & 0 \\ & & 1 \end{pmatrix}
\end{aligned}$$

where $s_{ij} = \text{sign}(a_{ij})$.

We are now ready to present our perturbation results.

THEOREM 3.3. *Let $A = [a_{ij}]$ and $\widetilde{A} = [\widetilde{a}_{ij}]$ be two symmetric positive semi-definite diagonally dominant matrices and let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $\widetilde{\lambda}_1 \le \widetilde{\lambda}_2 \le \cdots \le \widetilde{\lambda}_n$ be their eigenvalues respectively. If for some $0 \le \epsilon < 1$,*

$$|a_{ij} - \widetilde{a}_{ij}| \le \epsilon |a_{ij}|, \quad \textit{for all } i \ne j \tag{3.5}$$

and

$$|v_i - \widetilde{v}_i| \le \epsilon v_i, \quad \textit{for all } i \tag{3.6}$$

where $v_i = a_{ii} - \sum_{j\neq i} |a_{ij}|$ *and* $\widetilde{v}_i = \widetilde{a}_{ii} - \sum_{j\neq i} |\widetilde{a}_{ij}|$ *(i.e.,* $A = \mathcal{D}(A_D, v)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$ *are respectively the representations (2.1) of* A *and* $\widetilde{A}$ *with* $v = [v_i]$ *and* $\widetilde{v} = [\widetilde{v}_i]$*), then we have for all* i*,*

$$|\widetilde{\lambda}_i - \lambda_i| \leq \epsilon \lambda_i \tag{3.7}$$

Proof. Let D_i, L_i and $\widetilde{D}_i, \widetilde{L}_i$ be the matrices defined from A and $\widetilde{A}$ respectively according to (3.2) and (3.3). By (3.5), a_{ij} and $\widetilde{a}_{ij}$ have the same sign. Since L_i and $\widetilde{L}_i$ are defined from the signs of a_{ij} and $\widetilde{a}_{ij}$, we have

$$L_i = \widetilde{L}_i.$$

Furthermore, it follows from (3.5) and (3.6) that

$$(1-\epsilon)|a_{ij}| \leq |\widetilde{a}_{ij}| \leq (1+\epsilon)|a_{ij}|, \quad (1-\epsilon)v_i \leq \widetilde{v}_i \leq (1+\epsilon)v_i$$

where we note that $v_i \geq 0$ and $\widetilde{v}_i \geq 0$ by the assumption. This leads to

$$(1-\epsilon)D_i \leq \widetilde{D}_i \leq (1+\epsilon)D_i, \quad (1-\epsilon)V_0 \leq \widetilde{V}_0 \leq (1+\epsilon)V_0,$$

where $V_0 = \mathrm{diag}(v)$ and $\widetilde{V}_0 = \mathrm{diag}(\widetilde{v})$. Now, applying Lemma 3.2, we have

$$A = V_0 + L_1 D_1 L_1^T + \cdots + L_{n-1} D_{n-1} L_{n-1}^T$$

and

$$\widetilde{A} = \widetilde{V}_0 + L_1 \widetilde{D}_1 L_1^T + \cdots + L_{n-1} \widetilde{D}_{n-1} L_{n-1}^T.$$

Thus

$$(1-\epsilon)A \leq \widetilde{A} \leq (1+\epsilon)A, \tag{3.8}$$

from which it follows that $(1-\epsilon)\lambda_i \leq \widetilde{\lambda}_i \leq (1+\epsilon)\lambda_i$. The theorem is proved. □

Remark 1: From the assumptions (3.5) and (3.6), we have that $|\widetilde{a}_{ii} - a_{ii}| \leq \epsilon |a_{ii}|$, see [2]. The converse is not true, namely, $|\widetilde{a}_{ij} - a_{ij}| \leq \epsilon |a_{ij}|$ for all i, j does not imply $|v_i - \widetilde{v}_i| \leq \epsilon v_i$.

Remark 2: Our bound is sharp. This can be verified by considering a diagonal A.

Remark 3: We have assumed in the theorem that both A and $\widetilde{A}$ are symmetric positive semi-definite and diagonally dominant. However, we can assume only that A is symmetric positive semi-definite and diagonally dominant and $\widetilde{A}$ is symmetric and satisfies (3.6), which imply that $\widetilde{A}$ is also positive semi-definite and diagonally dominant, because it follows from (3.6) that $\widetilde{v}_i \geq 0$.

The above theorem can be easily generalized to the definite symmetric pencil eigenvalue problem $Ax = \lambda Bx$.

THEOREM 3.4. *Let* $A = [a_{ij}]$, $\widetilde{A} = [\widetilde{a}_{ij}]$ *be symmetric positive semi-definite diagonally dominant matrices and* $B = [b_{ij}]$ *and* $\widetilde{B} = [\widetilde{b}_{ij}]$ *be symmetric positive definite diagonally dominant matrices. Let* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ *and* $\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \cdots \leq \widetilde{\lambda}_n$ *be the eigenvalues of the pencil* $A - \lambda B$ *and* $\widetilde{A} - \lambda \widetilde{B}$ *respectively. Let* $A = \mathcal{D}(A_D, v)$, $B = \mathcal{D}(B_D, w)$, $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$ *and* $\widetilde{B} = \mathcal{D}(\widetilde{B}_D, \widetilde{w})$ *be their diagonal dominant part* *representations (2.1) with* $v = [v_i]$, $w = [w_i]$, $\widetilde{v} = [\widetilde{v}_i]$ *and* $\widetilde{w} = [\widetilde{w}_i]$. *If for some* $0 \le \epsilon, \epsilon' < 1$,

$$|a_{ij} - \widetilde{a}_{ij}| \le \epsilon |a_{ij}|, \quad |v_i - \widetilde{v}_i| \le \epsilon v_i,$$

and

$$|b_{ij} - \widetilde{b}_{ij}| \le \epsilon' |b_{ij}|, \quad |w_i - \widetilde{w}_i| \le \epsilon' w_i,$$

where $i \neq j$, *then for all* i

$$|\widetilde{\lambda}_i - \lambda_i| \le \frac{\epsilon + \epsilon'}{1 - \epsilon'} \lambda_i.$$

Proof. As in the proof of Theorem 3.3, we have the bound (3.8) for $\widetilde{A}$ and the following corresponding bound for $\widetilde{B}$

$$(1 - \epsilon')B \le \widetilde{B} \le (1 + \epsilon')B. \tag{3.9}$$

Now, using the minimax theorem, we obtain

$$\frac{1 - \epsilon}{1 + \epsilon'} \lambda_i \le \widetilde{\lambda}_i \le \frac{1 + \epsilon}{1 - \epsilon'} \lambda_i$$

which leads to the theorem. □

The theorems show that if the data $\mathcal{D}(A_D, v)$ representing A are known to a certain relative accuracy, then its eigenvalues are determined to the same relative accuracy. This is even true for the zero eigenvalue. For the smallest eigenvalue of a diagonally dominant M-matrix, our result improves the perturbation bound in [2], where only $|\lambda - \widetilde{\lambda}|/\lambda \le (2n - 1)\epsilon + O(\epsilon^2)$ is obtained. It can be applied then to the electronic circuit application as in [2] to obtain significantly improved perturbation bounds on the circuit speed. The improvement is achieved of course with the condition that the matrix is symmetric.

Acknowledgement: I would like to thank Prof. Jim Demmel for many valuable discussions. This work was inspired by an illuminating example in his book [4].

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