MA 522 Solution for Homework Assignment #2
(Based on the solutions of Megan Dailey)

Problem 1:

(1) The trace of a matrix $A = [a_{ij}]$ is defined as the sum of diagonal entries, i.e. $tr(A) = \sum_{i=1}^{n} a_{ii}$. Verify that $||A||_F^2 = tr(A^T A)$.

(2) If U, V are orthogonal, prove $||UAV||_F = ||A||_F$ and $||UAF||_2 = ||A||_2$.

<u>Proof:</u>

(1) Recall $(BA)_{ij} = \sum_{k=1}^{m} b_{ik}a_{kj}$ where $1 \leq i \leq p, 1 \leq j \leq n$ with $B = [b_{ij}] \in \mathbb{R}^{p\times m}, A = [a_{ij}] \in \mathbb{R}^{m\times n}$. Thus,

$$\begin{aligned} tr(BA) &= \sum_{i=1}^{n} (BA)_{ii} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik}a_{ki} \end{aligned}$$

Thus, if $B = A^T$, i.e. $b_{ik} = a_{ki}$, then

$$\begin{aligned} tr(A^T A) &= \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ki}a_{ki} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ki}^2 \\ &= ||A||_F^2 \end{aligned}$$

(2) Recall for an orthogonal matrix U,
  (a) $U^T = U^{-1}$
  (b) $||U\mathbf{x}||_2 = ||\mathbf{x}||_2$

Also, note

$$\begin{aligned} tr(AB) &= \sum_{k=1}^{m} \sum_{i=1}^{n} a_{ki}b_{ik} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik}a_{ki} \\ &= tr(BA) \end{aligned}$$

Now, consider

$$\begin{aligned}
||UAV||_F^2 &= tr\left((UAV)^T(UAV)\right) && \text{by (1)}\\
&= tr\left(V^TA^TU^TUAV\right)\\
&= tr\left(V^TA^TAV\right) && \text{by (a)}\\
&= tr\left((A^TAV)V^T\right) && \text{by } tr(AB)=tr(BA)\\
&= tr\left(A^TA\right) && \text{by (a)}\\
&= ||A||_F^2 && \text{by (1)}
\end{aligned}$$

And,

$$\begin{aligned}
||UAV||_2 &= \max_{\mathbf{x}\neq\mathbf{0}} \frac{||UAV\mathbf{x}||_2}{||\mathbf{x}||_2} && \text{by definition}\\
&= \max_{\mathbf{x}\neq\mathbf{0}} \frac{||U(AV\mathbf{x})||_2}{||\mathbf{x}||_2}\\
&= \max_{\mathbf{x}\neq\mathbf{0}} \frac{||AV\mathbf{x}||_2}{||\mathbf{x}||_2} && \text{by (b)}\\
&= \max_{\mathbf{x}\neq\mathbf{0}} \frac{||AV\mathbf{x}||_2}{||V\mathbf{x}||_2} && \text{by (b)}\\
&= \max_{\mathbf{y}\neq\mathbf{0}} \frac{||A\mathbf{y}||_2}{||\mathbf{y}||_2} && \text{where } \mathbf{y}=A\mathbf{x}\neq 0 if \mathbf{x}\neq 0\\
&= ||A||_2 && \text{by definition}
\end{aligned}$$

□

Problem 2: 1.7
Verify that $||xy^H||_F = ||xy^H||_2 = ||x||_2||y||_2$ for any $x, y \in \mathbb{C}^n$.

Proof:
Consider:

$$\begin{aligned}
||xy^H||_2 &= \max_{\mathbf{z}\neq\mathbf{0}} \frac{||(\mathbf{x}\mathbf{y}^H)\mathbf{z}||_2}{||\mathbf{z}||_2}\\
&= \max_{\mathbf{z}\neq\mathbf{0}} \frac{||\mathbf{x}(\mathbf{y}^H\mathbf{z})||_2}{||\mathbf{z}||_2}\\
&= \max_{\mathbf{z}\neq\mathbf{0}} \frac{||\mathbf{x}\langle\overline{\mathbf{y}},\mathbf{z}\rangle||_2}{||\mathbf{z}||_2}\\
&= \max_{\mathbf{z}\neq\mathbf{0}} \frac{|\langle\overline{\mathbf{y}},\mathbf{z}\rangle|\,||\mathbf{x}||_2}{||\mathbf{z}||_2}\\
&\leq \max_{\mathbf{z}\neq\mathbf{0}} \frac{||\overline{\mathbf{y}}||_2||\mathbf{z}||_2||\mathbf{x}||_2}{||\mathbf{z}||_2} && \text{by Cauchy Schwarz}\\
&= \max_{\mathbf{z}\neq\mathbf{0}} ||\overline{\mathbf{y}}||_2||\mathbf{x}||_2\\
&= ||\overline{\mathbf{y}}||_2||\mathbf{x}||_2\\
&= ||\mathbf{y}||_2||\mathbf{x}||_2 && \text{since } |\overline{y_i}|=|y_i|
\end{aligned}$$

Now let $\mathbf{z} = \frac{\overline{\mathbf{y}}}{||\overline{\mathbf{y}}||_2}$. Then, $||\mathbf{z}||_2 = 1$ and

$$\begin{aligned}
|\langle \overline{\mathbf{y}}, \mathbf{z} \rangle| &= \left| \left\langle \overline{\mathbf{y}}, \frac{\overline{\mathbf{y}}}{||\overline{\mathbf{y}}||_2} \right\rangle \right| \\
&= \frac{|\langle \overline{\mathbf{y}}, \overline{\mathbf{y}} \rangle|}{||\overline{\mathbf{y}}||_2} \\
&= \frac{||\overline{\mathbf{y}}||_2^2}{||\overline{\mathbf{y}}||_2} \\
&= ||\overline{\mathbf{y}}||_2
\end{aligned}$$

Thus,

$$\begin{aligned}
||xy^H||_2 &= \max_{\mathbf{z} \neq \mathbf{0}} \frac{|\langle \overline{\mathbf{y}}, \mathbf{z} \rangle| \, ||\mathbf{x}||_2}{||\mathbf{z}||_2} \\
&\geq \frac{||\overline{\mathbf{y}}||_2 ||\mathbf{x}||_2}{||\mathbf{z}||_2} && \text{from above} \\
&= ||\overline{\mathbf{y}}||_2 ||\mathbf{x}||_2 && \text{since } ||\mathbf{z}||_2 = 1 \\
&= ||\mathbf{y}||_2 ||\mathbf{x}||_2
\end{aligned}$$

Hence, we have

$$||x||_2 ||y||_2 \leq ||\mathbf{x}\mathbf{y}^H||_2 \leq ||x||_2 ||y||_2$$

Which implies,

$$||\mathbf{x}\mathbf{y}^H||_2 = ||x||_2 ||y||_2$$

Consider now

$$\begin{aligned}
||\mathbf{x}\mathbf{y}^H||_F^2 &= \sum_{j=1}^{n} \sum_{i=1}^{n} (x_i \overline{y_j})^2 \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} x_i^2 \overline{y_j}^2 \\
&= \sum_{i=1}^{n} x_i^2 \cdot \sum_{j=1}^{n} \overline{y_j}^2 \\
&= \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{j=1}^{n} \overline{y_j}^2 \right) \\
&= ||\mathbf{x}||_2^2 ||\overline{\mathbf{y}}||_2^2
\end{aligned}$$

Hence,

$$\begin{aligned}
||\mathbf{x}\mathbf{y}^H||_F &= ||\mathbf{x}||_2 ||\overline{\mathbf{y}}||_2 \\
&= ||\mathbf{x}||_2 ||\mathbf{y}||_2
\end{aligned}$$

Thus,

$$||\mathbf{x}\mathbf{y}^H||_2 = ||x||_2 ||y||_2 = ||\mathbf{x}\mathbf{y}^H||_F$$

□

Problem 3: Let k be a postive integer and $X \in \mathbb{R}^{n \times n}$ be such that $X^k = \mathbf{0}$. Prove that $I - X$ is invertible and

$$(I - X)^{-1} = \sum_{i=0}^{k-1} X^i$$

Proof:
Since

$$\begin{aligned}
(I - X) \cdot \sum_{i=0}^{k-1} X^i &= \sum_{i=0}^{k-1} X^i - X \cdot \sum_{i=0}^{k-1} X^i \\
&= \sum_{i=0}^{k-1} X^i - \sum_{i=0}^{k-1} X^{i+1} \\
&= X^0 - X^k \\
&= I - \mathbf{0} && \text{by definition and assumption} \\
&= I
\end{aligned}$$

Hence, $(I - X)$ is invertible and moreover $(I - X)^{-1} = \sum_{i=0}^{k-1} X^i$.

□

Problem 4: Show that if $||E|| < 1/2$, then

$$||\,(I - E)^{-1} - (I + E)\,|| \leq 2||E||^2$$

where $||\cdot||$ is any matrix operator norm.

Proof:
Remarks:
(a) For $|r| < 1$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1 - r}$$

So,

$$\begin{aligned}
\sum_{i=2}^{\infty} r^i &= \sum_{i=0}^{\infty} r^i - 1 - r \\
&= \frac{1}{1 - r} - (1 + r) \\
&= \frac{1}{1 - r} - \frac{(1 + r)(1 - r)}{1 - r} \\
&= \frac{r^2}{1 - r}
\end{aligned}$$

(b) Consider

$$\begin{aligned} ||E|| &< \frac{1}{2} \\ -||E|| &> -\frac{1}{2} \\ 1 - ||E|| &> \frac{1}{2} \\ \frac{1}{1 - ||E||} &< 2 \end{aligned}$$

Recall from class:

$$(I - E)^{-1} = \sum_{i=0}^{\infty} E^i$$

Thus, we have

$$(I - E)^{-1} - (I - E) = \sum_{i=2}^{\infty} E^i$$

Hence,

$$\begin{aligned} ||\,(I - E)^{-1} - (I - E)\,|| &= ||\sum_{i=2}^{\infty} E^i|| \\ &\leq \sum_{i=2}^{\infty} ||E^i|| && \text{by the triangle inequality} \\ &= \frac{||E||^2}{1 - ||E||} && \text{by Remark (a)} \\ &\leq 2||E||^2 && \text{by Remark (b)} \end{aligned}$$

□