MA 522 Solution for Homework Assignment #3
(Based on the solutions of John Rotramel)

1. Assume that A is invertible and $||A^{-1}||||\delta A|| < 1$. Prove that

$$\frac{||(A+\delta A)^{-1} - A^{-1}||}{||A^{-1}||} \le \frac{\kappa(A)\frac{||\delta A||}{||A||}}{1-\kappa(A)\frac{||\delta A||}{||A||}}.$$

($||\cdot||$ is any matrix operator norm.)

Proof: $||(A+\delta A)^{-1} - A^{-1}|| = ||[AA^{-1}(A+\delta A)]^{-1} - A^{-1}|| =$
$||[A(I + A^{-1}\delta A)]^{-1} - A^{-1}|| = ||(I + A^{-1}\delta A)^{-1}A^{-1} - A^{-1}|| =$
$||[(I + A^{-1}\delta A)^{-1} - I]A^{-1}|| \le ||(I + A^{-1}\delta A)^{-1} - I||||A^{-1}||$, so that

$$\frac{||(A+\delta A)^{-1} - A^{-1}||}{||A^{-1}||} \le ||(I + A^{-1}\delta A)^{-1} - I||.$$

Then
$||(I + A^{-1}\delta A)^{-1} - I|| = || - I + (I - (-A^{-1}\delta A)^{-1})|| = || - I + \sum_{i=0}^{\infty}(-A^{-1}\delta A)^i|| =$
$|| - I + I + \sum_{i=1}^{\infty}(-A^{-1}\delta A)^i|| \le \sum_{i=1}^{\infty}||(-A^{-1}\delta A)^i|| \le \sum_{i=1}^{\infty}|| - A^{-1}\delta A||^i = \frac{||A^{-1}\delta A||}{1-||A^{-1}\delta A||}$.

Since $||A^{-1}\delta A|| \le ||A^{-1}||||\delta A|| \Longrightarrow 1 - ||A^{-1}\delta A|| \ge 1 - ||A^{-1}||||\delta A|| \Longrightarrow$
$\frac{1}{1-||A^{-1}\delta A||} \le \frac{1}{1-||A^{-1}||||\delta A||}$, we get

$$\frac{||(A+\delta A)^{-1} - A^{-1}||}{||A^{-1}||} \le \frac{||A^{-1}\delta A||}{1-||A^{-1}\delta A||} \le \frac{||A^{-1}||||\delta A||}{1-||A^{-1}||||\delta A||} = \frac{||A^{-1}||\frac{||A||}{||A||}||\delta A||}{1-||A^{-1}||\frac{||A||}{||A||}||\delta A||} = \frac{\kappa(A)\frac{||\delta A||}{||A||}}{1-\kappa(A)\frac{||\delta A||}{||A||}}$$

□

2. If $A = BC$ and A is invertible, where A, B, C are square matrices, prove

$$\kappa(A) \leq \kappa(B)\kappa(C).$$

If B is an orthogonal matrix, prove

$$\kappa_2(A) = \kappa_2(C).$$

Proof: Since A is invertible, (BC) is invertible, and $A^{-1} = (BC)^{-1} = C^{-1}B^{-1}$. Then $||A|| = ||BC|| \leq ||B||||C||$ and $||A^{-1}|| = ||(BC)^{-1}|| = ||C^{-1}B^{-1}|| \leq ||C^{-1}||||B^{-1}||$. Combining the two inequalities and applying the definition of condition number yields

$$\kappa(A) = ||A||||A^{-1}|| \leq ||C^{-1}||||B^{-1}||||B||||C|| = ||B||||B^{-1}||||C||||C^{-1}|| = \kappa(B)\kappa(C).$$

For B orthogonal, $||BC||_2 = ||C||_2$, by HW 2 - Problem 1, yielding $||A||_2 = ||C||_2$. Next, as $B^{-1} = B^T$ is also orthogonal. $||A^{-1}||_2 = ||C^{-1}B^{-1}||_2 = ||C^{-1}||_2$. Combining the above results gives

$$\kappa_2(A) = ||A||_2||A^{-1}||_2 = ||C||_2||C^{-1}||_2 = \kappa_2(C).$$

□

3. A matrix of the form $G = I - ge_k^T$ (where $g \in \mathbb{R}^n$ and $e_k \in \mathbb{R}^n$ is the k-th coordinate vector) is called a Gauss-Jordan matrix.

1. Given a x with $e_k^T x \neq 0$, show that there exists a Gauss-Jordan matrix G s.t. Gx is a multiple of e_k.
2. Given an $n \times n$ matrix A, construct Gauss-Jordan matrices $G_1, G_2, \cdots, G_n$ successively such that

$$G_n \cdots G_2 G_1 A \text{ is diagonal}$$

Solution (1):
For $G\mathbf{x}$ to be a multiple α of $\mathbf{e}_k$, it must be true that $\alpha\mathbf{e}_k = G\mathbf{x} = (I - \mathbf{g}\mathbf{e}_k^T)\mathbf{x} = \mathbf{x} - \mathbf{g}\mathbf{e}_k^T\mathbf{x} = \mathbf{x} - \mathbf{g}(\mathbf{e}_k^T\mathbf{x}) = \mathbf{x} - \mathbf{g}(x_k)$, since the product $\mathbf{e}_k^T\mathbf{x}$ is a scalar, the k^{th} element of $\mathbf{x}$, and is nonzero by hypothesis. Thus

$$-\frac{1}{x_k}(\alpha\mathbf{e}_k - \mathbf{x}) = \frac{1}{x_k}(\mathbf{x} - \alpha\mathbf{e}_k) = \mathbf{g}$$

will produce the desired G :
$G\mathbf{x} = (I - \mathbf{g}\mathbf{e}_k^T)\mathbf{x} = [I - \frac{1}{x_k}(\mathbf{x} - \alpha\mathbf{e}_k)\mathbf{e}_k^T]\mathbf{x} = \mathbf{x} - \frac{1}{x_k}(\mathbf{x} - \alpha\mathbf{e}_k)x_k = \mathbf{x} - \mathbf{x} + \alpha\mathbf{e}_k = \alpha\mathbf{e}_k$. □

Solution (2):
In order to convert the k^{th} column of A to all zeros except for the k^{th} element (the one on the main diagonal), the $\mathbf{g}$ vector must be like the t_k vectors used to generate M_k matrices, except the upper triangular entries (those with indices less than k) are not necessarily zero:
$\mathbf{g}_k^T = [\frac{a_{1k}^{(k-1)}}{a_{kk}^{(k-1)}} \ \frac{a_{2k}^{(k-1)}}{a_{kk}^{(k-1)}} \ \ldots \ \frac{a_{k-1,k}^{(k-1)}}{a_{kk}^{(k-1)}} \ 0 \ \frac{a_{k+1,k}^{(k-1)}}{a_{kk}^{(k-1)}} \ldots \ \frac{a_{nk}^{(k-1)}}{a_{kk}^{(k-1)}}]$,
where $a_{ik}^{(k-1)}, 1 \leq i \leq n$, is an element from $A_{k-1} = G_{k-1} \ldots G_1 A$, and $a_{ik}^{(0)}$ is an element from A. $a_{kk}^{(k-1)}$ cannot be zero. This gives

$$G_k = I - \mathbf{g}_k\mathbf{e}_k^T = \begin{bmatrix} 1 & 0 & \ldots & 0 & -\frac{a_{1k}^{(k-1)}}{a_{kk}^{(k-1)}} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & -\frac{a_{2k}^{(k-1)}}{a_{kk}^{(k-1)}} & 0 & \ldots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 & -\frac{a_{k-1,k}^{(k-1)}}{a_{kk}^{(k-1)}} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & -\frac{a_{k+1,k}^{(k-1)}}{a_{kk}^{(k-1)}} & 1 & \ldots & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 & -\frac{a_{nk}^{(k-1)}}{a_{kk}^{(k-1)}} & 0 & \ldots & 1 \end{bmatrix}.$$

□