MA 522 Solution for Homework Assignment #3
(Based on the solutions of John Rotramel)

1. (Question 2.13, Parts 1 and 2, from the text) In this question we will ask how to solve $By = c$ given a fast way to solve $Ax = b$, where $A - B$ is "small" in some sense.

1. Prove the *Sherman-Morrison formula:* Let A be nonsingular, u and v be column vectors, and $A+uv^T$ be nonsingular. Then $(A+uv^T)^{-1} = A^{-1} - (A^{-1}uv^TA^{-1})/(1+v^TA^{-1}u)$.

   More generally, prove the *Sherman-Morrison-Woodbury formula:* Let U and V be n-by-k rectangular matrices, where $k \le n$ and A is n-by-n. Then $T = I+V^TA^{-1}U$ is nonsingular if and only if $A+UV^T$ is nonsingular, in which case $(A+UV^T)^{-1} = A^{-1} - A^{-1}UT^{-1}V^TA^{-1}$.
2. If you have a fast algorithm to solve $Ax = b$, show how to build a fast solver for $By = c$, where $B = A + uv^T$.

Proof (Sherman-Morrison formula):
Since $(A + uv^T)$ is nonsingular, $(A + uv^T)^{-1}$ exists, and the product of the two is I. Because $v^TA^{-1}u$ multiplies to form a scalar, the denominator of the fraction is scalar. So letting $\omega = v^TA^{-1}u$ and multiplying yields

$$(A + uv^T)\left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}\right) = (A + uv^T)\left(\frac{A^{-1} + A^{-1}v^TA^{-1}u - A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}\right) =$$

$$\frac{AA^{-1} + uv^TA^{-1} + AA^{-1}v^TA^{-1}u + uv^TA^{-1}v^TA^{-1}u - AA^{-1}uv^TA^{-1} - uv^TA^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u} =$$

$$\frac{I + uv^TA^{-1} + I\omega + uv^TA^{-1}\omega - Iuv^TA^{-1} - u\omega v^TA^{-1}}{1 + \omega} =$$

$$\frac{I(1 + \omega) + uv^TA^{-1} + \omega uv^TA^{-1} - uv^TA^{-1} - \omega uv^TA^{-1}}{1 + \omega} = \frac{I(1 + \omega)}{1 + \omega} = I.$$

Therefore the Sherman-Morrison formula holds. □

Proof (Sherman-Morrison-Woodbury formula):
$(\implies)$ If $T = I + V^TA^{-1}U$ is nonsingular, then

$$(A + UV^T)(A^{-1} - A^{-1}UT^{-1}V^TA^{-1}) =$$

$$\mathrm{AA}^{-1} + UV^TA^{-1} - AA^{-1}UT^{-1}V^TA^{-1} - UV^TA^{-1}UT^{-1}V^TA^{-1} =$$
$$\mathrm{I{+}UV}^TA^{-1} - UT^{-1}V^TA^{-1} - UV^TA^{-1}UT^{-1}V^TA^{-1} =$$
$$\mathrm{I{+}U(V}^TA^{-1}) - UT^{-1}(V^TA^{-1}) - UV^TA^{-1}UT^{-1}(V^TA^{-1}) =$$
$$\mathrm{I{+}U(I{-}T}^{-1} - V^TA^{-1}UT^{-1})(V^TA^{-1}) = I + U(I - T^{-1}(I + V^TA^{-1}U))(V^TA^{-1})$$
$$=\mathrm{I{+}U(I{-}T}^{-1}(I + V^TA^{-1}U))(V^TA^{-1})$$
$$=\mathrm{I{+}U(I{-}T}^{-1}T)(V^TA^{-1}) = I.$$

Therefore, $(A + UV^T)$ is invertible.

$(\Longleftarrow)$ If $(A + UV^T)$ is nonsingular with the given inverse, and using the hint from the class website to verify $T^{-1} = I - V^T(A + UV^T)^{-1}U$, we have

$T(I - V^T(A + UV^T)^{-1}U) = (I + V^TA^{-1}U)(I - V^T(A + UV^T)^{-1}U) =$
I+V$^TA^{-1}U - V^T(A + UV^T)^{-1}U - V^TA^{-1}UV^T(A + UV^T)^{-1}U =$
I+V$^TA^{-1}U - V^T[(A + UV^T)^{-1} + A^{-1}UV^T(A + UV^T)^{-1}]U =$
I+V$^TA^{-1}U - V^T[(I + A^{-1}UV^T)(A + UV^T)^{-1}]U =$
I+V$^TA^{-1}U - V^T[A^{-1}(A + UV^T)(A + UV^T)^{-1}]U = I + V^TA^{-1}U - V^TA^{-1}U = I$
So that T is invertible. □

Solution (2):
Given a fast solver for $Ax = b$, we first solve $Ay = c$ and $Az = u$. From y and z, we construct the solution of $Bx = c$ as

$$x = y - \frac{v^Ty}{1 + v^Tz}z$$

because

$$\begin{aligned} x &= B^{-1}c && (1)\\ &= A^{-1}c - (A^{-1}uv^TA^{-1}c)/(1 + v^TA^{-1}u) && (2)\\ &= y - z(v^Ty)/(1 + v^Tz) && (3) \end{aligned}$$

□

2. Let $\hat{X} = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n]$ be the computed inverse of A where $\hat{x}_i$ is obtained by solving $Ax_i = e_i$ using the computed LU factorization of A (e_i is the i-th column of I). Use the bound on the residual of $\hat{x}_i$ to show that

$$A\hat{X} = I + F$$

with $||F||_\infty \leq 3g_{pp}n^4\epsilon||A||_\infty||\hat{X}||_\infty + O(\epsilon^2)$.

Proof: The residual is $\mathbf{r} = A\hat{\mathbf{x}} - \mathbf{b}$, so here, for $F = [f_1 \ f_2 \ \cdots \ f_n]$ we have $A\hat{X} = I + F \implies A\hat{x}_i = e_i + f_i \implies f_i = A\hat{x}_i - e_i$, so that the f_i represent the residuals for each column.

By theorem, $||\delta A||_\infty \leq (3n\epsilon + O(\epsilon^2))||L||_\infty||U||_\infty \leq (3n\epsilon + O(\epsilon^2))(n)(n||U||_{\max})\frac{||A||_\infty}{||A||_{\max}} \leq (3n\epsilon + O(\epsilon^2))g_{pp}n^2||A||_\infty$.

Comparing $(A + \delta A)\hat{x}_i = e_i \implies A\hat{x}_i + \delta A\hat{x}_i = e_i \implies A\hat{x}_i = e_i - \delta A\hat{x}_i$ with $A\hat{x}_i = e_i + f_i$ yields $-f_i = \delta A\hat{x}_i$. Thus

$$||f_i||_\infty = ||\delta A\hat{x}_i||_\infty \leq ||\delta A||_\infty||\hat{x}_i||_\infty \leq (3n\epsilon + O(\epsilon^2))g_{pp}n^2||A||_\infty||\hat{x}_i||_\infty.$$

Then, where k is the row index of F or X,
$||F||_\infty = \max_{1\leq k\leq n}\sum_{i=1}^n |f_{ki}| \leq \sum_{i=1}^n \max_{1\leq k\leq n}|f_{ki}| \leq \sum_{i=1}^n ||f_i||_\infty \leq \sum_{i=1}^n (3n\epsilon + O(\epsilon^2))g_{pp}n^2||A||_\infty||\hat{x}_i||_\infty = (3n\epsilon + O(\epsilon^2))g_{pp}n^2||A||_\infty\sum_{i=1}^n ||\hat{x}_i||_\infty$.
But $\sum_{i=1}^n ||\hat{x}_i||_\infty \leq \sum_{i=1}^n ||\hat{X}||_\infty = n||\hat{X}||_\infty$, yielding

$$||F||_\infty \leq (3n\epsilon + O(\epsilon^2))g_{pp}n^2||A||_\infty n||\hat{X}||_\infty = 3n^4g_{pp}\epsilon||A||_\infty||\hat{X}||_\infty + O(\epsilon^2)n^3g_{pp}||A||_\infty||\hat{X}||_\infty.$$

Subsuming the coefficients into "order" (which works here since $n^3 < n^4$) yields

$$||F||_\infty \leq 3g_{pp}n^4\epsilon||A||_\infty||\hat{X}||_\infty + O(\epsilon^2).$$

□