MA 522 - Midterm Exam

Oct 22, 2008

Problem 1. (7 points)

1. Let A and B be $n \times n$ matrices and $\|\cdot\|$ is a matrix operator norm. Use the definition of the operator norm to prove $\|AB\| \leq \|A\| \cdot \|B\|$.

2. Let Q be an $m \times m$ orthogonal matrix and let A be an $m \times n$ matrix. Prove $\|QA\|_2 = \|A\|_2$.

Problem 2. (8 points) Let A be an $n \times n$ invertible matrix and δA be an $n \times n$ matrix such that $\kappa(A)\frac{\|\delta A\|}{\|A\|} < 1$. Prove that $A + \delta A$ is invertible and

$$\frac{\|(A+\delta A)^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \frac{\kappa(A)\frac{\|\delta A\|}{\|A\|}}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}$$

where $\|\cdot\|$ is any matrix operator norm and $\kappa(A)$ is the condition number of A.

Problem 3. (7 points) Let U be a nonsingular upper triangular matrix (the diagonals need not be ones). Let $\hat{x}$ be the computed solution to $Ux = y$ using backward substitution. Prove that $(U + \delta U)\hat{x} = y$ with $|\delta U| \leq (n\epsilon + O(\epsilon^2))|U|$
(You may use the fact that $fl(\sum_{i=1}^{d} x_i y_i) = \sum_{i=1}^{d} x_i y_i (1 + \delta_i)$ with $\delta_i \leq d\epsilon + \mathcal{O}(\epsilon^2)$.)

Problem 4. (8 points) Let $A \in R^{n \times n}$ be a symmetric positive definite matrix.

1. Write down the Cholesky Algorithm for computing the Cholesky factorization $A = GG^T$
2. Prove that $|g_{ij}| \leq \sqrt{a_{ii}}$ for any $1 \leq i \leq j \leq n$, where $G = [g_{ij}]$.