Research Statement
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The research I have done thus far has been in finite solvable group theory. More specifically, I have studied and characterized several classes of finite solvable $\mathcal{T}$-groups. Let me first introduce some definitions, notation, and known theorems before discussing my results.

A subgroup H is said to be permutable (Sylow-permutable) in G if it satisfies the property $HK = KH$ for all subgroups (Sylow subgroups) K of G. Among the first to study permutable (Sylow-permutable) subgroups was O. Ore [8] (O. Kegel [7]) who showed such groups are necessarily subnormal[1]. It is clear from the definitions that normal subgroups are permutable and permutable subgroups are Sylow-permutable. $H \trianglelefteq G$ ($H \operatorname{per} G$, $H \operatorname{S-per} G$) denotes H is normal (permutable, Sylow-permutable) in G.

A $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group) is a group G which possesses a transitive normality (permutability, Sylow-permutability) relation. That is, G is a $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group) if for all subgroups H and K of G where $H \trianglelefteq K \trianglelefteq G$ ($H \operatorname{per} K \operatorname{per} G$, $H \operatorname{S-per} K \operatorname{S-per} G$) we have $H \trianglelefteq G$ ($H \operatorname{per} G$, $H \operatorname{S-per} G$). Since normal, permutable, and Sylow-permutable subgroups are subnormal, it is a direct consequence that $\mathcal{T}$-groups ($\mathcal{PT}$-groups, $\mathcal{PST}$-groups) are precisely those groups in which subnormality and normality (permutability, Sylow-permutability) coincide. The theory of $\mathcal{T}$-groups ($\mathcal{PT}$-groups) probably began with a result of Dedekind [2] (Iwasawa [6]) which characterizes finite groups all of whose subgroups are normal (permutable). Such a group, called a Dedekind group (Iwasawa group), is clearly a $\mathcal{T}$-group ($\mathcal{PT}$-group). In the years 1953, 1964, and 1975, Gaschütz, Zacher, and Agrawal, respectively, proved the following definitive results on finite solvable $\mathcal{T}$-groups, $\mathcal{PT}$-groups, and $\mathcal{PST}$-groups.

Theorem 1 (Gaschütz [5], Zacher [10], Agrawal [1]) *Let G be a finite group with L the nilpotent residual[2] of G. Then G is a solvable $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group) if and only if the following conditions hold:*

1. *L is a normal abelian Hall[3] subgroup of G with odd order;*
2. *G/L is a Dedekind (Iwasawa, nilpotent) group.*
3. *Every subgroup of L is normal in G;*

For a class of groups, say $\mathcal{X}$, call any group $G \in \mathcal{X}$ an $\mathcal{X}$-group. Define the class $\mathrm{Hall}_{\mathcal{X}}$ to be the class of groups G possessing a normal nilpotent subgroup N such that G/N' is an $\mathcal{X}$-group, where N' is the derived subgroup of N. Define $\mathcal{T}_{\mathrm{o}}$-group to be the class of groups G where $G/\Phi(G)$[4] is a $\mathcal{T}$-group. I have been able to characterize, both locally and globally, finite solvable $\mathcal{T}_{\mathrm{o}}$-groups, and finite solvable $\mathrm{Hall}_{\mathcal{X}}$-groups where $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. It should be noted that the classes $\mathcal{T}_{\mathrm{o}}$ and $\mathrm{Hall}_{\mathcal{T}}$ were introduced in [3] and [9], respectively. However, no characterizations of the classes were given. There is no mention of the classes $\mathrm{Hall}_{\mathcal{PT}}$ and $\mathrm{Hall}_{\mathcal{PST}}$ in the current literature. The characterization theorems are as follows:

Theorem 2 *Let G be a finite solvable group with nilpotent residual L. G is a $\mathcal{T}_{\mathrm{o}}$-group if and only if G satisfies the following:*

1. *L is a nilpotent Hall subgroup of G of odd order;*
2. *$G/L\Phi(G)$ is a $\mathcal{T}$-group;*
3. *Every subgroup of $L\Phi(G)/\Phi(G)$ is normal in $G/\Phi(G)$.*

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[1] H is subnormal in G if there is a chain, $H \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n \trianglelefteq G$, of normal subgroups from H to G.
[2] L, the nilpotent residual of G, is the unique smallest subgroup of G such that G/L is nilpotent.
[3] L is Hall in G if the index of L in G is relatively prime to $|L|$. Hall subgroups are generalizations of Sylow subgroups.
[4] $\Phi(G)$, the Frattini subgroup of G, is defined to be the intersection of all maximal subgroups in G.

Theorem 3 *Let G be a finite solvable group with F the Fitting subgroup*[5] *of G, L the nilpotent residual of G, and $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. G is a $\mathrm{Hall}_{\mathcal{X}}$-group if and only if G satisfies the following:*

1. *L is a nilpotent Hall subgroup of G of odd order;*
2. *G/LF' is an $\mathcal{X}$-group;*
3. *Every subgroup of LF'/F' is normal in G/F'.*

It is clear from the definitions that $\mathcal{T} \subseteq \mathcal{PT} \subseteq \mathcal{PST}$, $\mathrm{Hall}_{\mathcal{T}} \subseteq \mathrm{Hall}_{\mathcal{PT}} \subseteq \mathrm{Hall}_{\mathcal{PST}}$, and that $\mathcal{X} \subseteq \mathrm{Hall}_{\mathcal{X}}$ for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. I have computed examples, with the use of the software package GAP [4] (Groups, Algorithms, and Programming), showing that the above inclusions are proper. However, we have the following theorem showing the relationship of the above groups with the class of finite solvable $\mathcal{T}_{\mathrm{o}}$-groups.

Theorem 4 *In the class of finite solvable groups, the classes $\mathrm{Hall}_{\mathcal{PST}}$ and $\mathcal{T}_{\mathrm{o}}$ are the same.*

This gives rise to the following diagram in the realm of finite solvable groups:

$$\begin{array}{ccccccc}
\mathcal{T} & \Longrightarrow & \mathcal{PT} & \Longrightarrow & \mathcal{PST} & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\mathrm{Hall}_{\mathcal{T}} & \Longrightarrow & \mathrm{Hall}_{\mathcal{PT}} & \Longrightarrow & \mathrm{Hall}_{\mathcal{PST}} & \Longleftrightarrow & \mathcal{T}_o
\end{array}$$

There has been much work done on local characterizations of finite solvable $\mathcal{T}$-groups, $\mathcal{PT}$-groups, and $\mathcal{PST}$-groups. I have found local characterizations of finite solvable $\mathcal{T}_{\mathrm{o}}$-groups and $\mathrm{Hall}_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. An example of one such characterization is:

Theorem 5 *Let $\mathscr{Y}_p$ be the class of groups G satisfying the property that, whenever $H \leq S \leq P \in \mathrm{Syl}_p(G)$, we have H S-per $N_G(S)$. Let ${F_p}'$ be the Sylow p-subgroup of the derived subgroup of the Fitting subgroup of a finite solvable group G. G is a finite solvable $\mathrm{Hall}_{\mathcal{PST}}$-group if and only if $G/{F_p}' \in \mathscr{Y}_p$ for all primes p.*

Other work I have done deals with the question of subgroup closure. Subgroups of finite solvable $\mathcal{T}$-groups are finite solvable $\mathcal{T}$-groups. A similar statement can be said about $\mathcal{PT}$-groups and $\mathcal{PST}$-groups, however, similar statements cannot be said about $\mathcal{T}_{\mathrm{o}}$-groups and $\mathrm{Hall}_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. One wonders what kind of group you would get if you force subgroup closure on $\mathcal{T}_{\mathrm{o}}$-groups and $\mathrm{Hall}_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. I have established the following:

Theorem 6 *The following are equivalent for a finite group G:*

1. *G is a solvable $\mathcal{PST}$-group;*
2. *G is a subgroup-closed $\mathrm{Hall}_{\mathcal{PST}}$-group;*
3. *G is a subgroup-closed $\mathcal{T}_{\mathrm{o}}$-group;*

Unfortunately, it is not the case that solvable $\mathcal{T}$-groups are precisely the subgroup-closed $\mathrm{Hall}_{\mathcal{T}}$-groups, or that solvable $\mathcal{PT}$-groups are precisely the subgroup-closed $\mathrm{Hall}_{\mathcal{PT}}$-groups. Counterexamples have been found with the use of GAP.

Aside from the research on my thesis, I have been working with Adolfo Ballester-Bolinches and Ramon Esteban-Romero, both of Valencia, Spain. We have discovered a new characterization of finite (both solvable and non-solvable) $\mathcal{PST}$-groups. We hope to publish these results in the near future.

Listed below are several other avenues for further related research.

- Characterizing subgroup-closed $\mathrm{Hall}_{\mathcal{T}}$ and $\mathrm{Hall}_{\mathcal{PT}}$-groups.
- Exploring the different classes of groups discussed in the non-solvable realm or even the infinite realm.
- Studying minimal-non-$\mathcal{X}$-groups, that is, groups which are not $\mathcal{X}$-groups, yet each proper subgroup is an $\mathcal{X}$-group, where $\mathcal{X}$ takes on the role of the different classes of groups discussed in my thesis.

[5] The Fitting subgroup of G is defined to be the group generated by all normal nilpotent subgroups of G.

- Studying subgroups permuting with all system normalizers of a finite solvable group.

Let me end by saying that this research has been challenging and rewarding. I look forward to continuing my research in group theory or other areas, either jointly or by myself. I am open to all possibilities.

References

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