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The mixed problem

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We consider the mixed boundary value problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \rho} = f_N & \text{on } N \\ (\nabla u)^* \in L^2(d\sigma) \end{cases}$$

where $(\nabla u)^*$ is the non-tangential maximal function. Recall that if v is defined in Ω and $P \in \partial\Omega$, we may define the non-tangential maximal function by

$$v^*(P) = \sup\{|v(x)| : |x - P| < 2 dist(x, \partial\Omega)\}.$$

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The domain Ω will be an open set with connected Lipschitz boundary.
The boundary is the union of two disjoint sets $\partial\Omega = D \cup N$ with D open. Additional conditions will be imposed.

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The Lamé system

We will consider several elliptic operators of the form

$$(Lu)^\alpha = \sum_{j=1}^{n} \frac{\partial}{\partial x_i} a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j}, \qquad \alpha = 1, \dots, n \qquad (1)$$

The Lamé system arises when

$$a^{ij}_{\alpha\beta} = \mu\delta_{ij}\delta_{\alpha\beta} + \lambda\delta_{i\alpha}\delta_{j\beta} + \mu\delta_{i\beta}\delta_{j\alpha}.$$

The constants $\mu > 0$ and $\lambda \geq 0$ describe the elastic properties of the material.

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The boundary operator

- We will occasionally use the convention that repeated indices are summed.
- We use ν for the unit outer normal to $\partial\Omega$.
- On N, we will use the boundary operator defined by

$$\left(\frac{\partial u}{\partial \rho}\right)^{\alpha} = \nu_i a^{ij}_{\alpha\beta} \frac{\partial u^{\beta}}{\partial x_j}.$$

Some history-Laplace's equation

- Solutions exist in the Sobolev space $H^1(\Omega)$ under quite general assumptions.
- The L^2 result fails for the Laplacian in smooth domains. The examples are quite simple, $u = \mathrm{Re}\sqrt{z}$ in a half-plane.
- The study of the mixed problem in Lipschitz domains was a problem posed by Kenig in his CBMS lecture notes (1991).
- A positive result was obtained by G. Savaré (1997) who showed that a solution could be found in the Besov space $B^{3/2,2}_{\infty}(\Omega)$ of smooth domains.

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More history

- In 1996, the speaker obtained a result for the Laplacian in a class of Lipschitz domains. Today's talk extends this method to the Lamé system.
- J. Sykes (Ph.D. 1999, published in 2002) obtains L^p results for the mixed problem for the Laplacian.
- I. Mitrea and M. Mitrea (2007) obtain results for the Laplacian in a large family of Sobolev and Besov spaces.
- There is a large literature on problems on polygonal and polyhedral domains that I do not attempt to summarize.
- The study of the Dirichlet and traction problems for the Lamé system was carried out by Dahlberg, Kenig and Verchota in 1988.

Creased domains

Joint work with I. Mitrea.

The boundary $\partial\Omega$ is locally the graph of a Lipschitz function, ϕ.

In addition, we require that if P is a point in $\bar{D} \cap \bar{N}$, then we have a coordinate system $(x', x_n) = (x_1, x'', x_n) \in \mathbf{R} \times \mathbf{R}^{n-2} \times \mathbf{R}$, and a second Lipschitz function $\psi : \mathbf{R}^{n-2} \to \mathbf{R}$ so that in some ball $B_r(P)$,

$$\begin{aligned} D \cap B_r(P) &= \{(x_1, x'', x_n) : x_n = \phi(x'),\ x_1 < \psi(x'')\} \cap B_r(P) \\ N \cap B_r(P) &= \{(x_1, x'', x_n) : x_n = \phi(x'),\ x_1 \geq \psi(x'')\} \cap B_r(P) \end{aligned}$$

We assume that there is a positive $\delta > 0$ so that $\phi_{x_1} \geq \delta$ if $x_1 > \psi(x'')$ and $\phi_{x_1} \leq -\delta$ if $x_1 \leq \psi(x'')$.

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How do we use the creased condition?

In a creased domain, we may find a smooth vector-valued function h and $\delta > 0$ so that

$$\begin{aligned} h \cdot \nu &> \delta \qquad \text{a.e. on } N \\ h \cdot \nu &< -\delta \qquad \text{a.e. on } D \end{aligned} \tag{2}$$

To do this, choose the unit vector in the e_1-direction in each coordinate cylinder and then patch these together with a smooth, non-negative partition of unity.
The explicit form of the crease is also useful in some homotopy arguments that are suppressed because this talk is already too long.

Ellipticity

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Our coefficients $a^{ij}_{\alpha\beta}$ will satisfy the ellipticity condition uniformly in r,

$$a^{ij}_{\alpha\beta}\xi^i_\alpha\xi^j_\beta \geq \mu\left|\frac{\xi+\xi^t}{2}\right|^2, \qquad \text{if } 0 \leq r \leq \mu, \ \lambda \geq 0. \qquad (3)$$

In these definitions, we define the norm of a matrix by

$$|\xi|^2 = \xi^\alpha_i \xi^\alpha_i \alpha.$$

Our coefficients will satisfy the symmetry condition

$$a^{ij}_{\alpha\beta} = a^{ji}_{\beta\alpha}. \qquad (4)$$

The Rellich-(Payne-Weinberger-Pohozaev-....) identity

Lemma (Dahlberg, Kenig and Verchota)

Let $h : \bar{\Omega} \to \mathbf{R}^m$ be a smooth vector field, suppose that the coefficients, $a^{ij}_{\alpha\beta}$, satisfy the symmetry condition (4), the ellipticity condition (3) and let u be a solution of $Lu = 0$ with $(\nabla u)^* \in L^2(\partial\Omega)$. We have the identity,

$$\int_{\partial\Omega} h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2\nu_i a^{ij}_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} h_k \frac{\partial u^\alpha}{\partial x_k} \, d\sigma$$

$$= \int_{\Omega} \frac{\partial h_k}{\partial x_k} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} - 2 \frac{\partial h_k}{\partial x_i} a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_k} \frac{\partial u^\beta}{\partial x_j} \, dx.$$

Either the proof is a direct application of the divergence theorem or there is a typo in the statement.

Korn inequality

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Let $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ be the strain.

Lemma (Korn's inequality)

If $D \subset \partial\Omega$ is of positive measure, $\partial\Omega$ is Lipschitz, then

$$\int_\Omega |u|^2 + |\nabla u|^2 \, dx \le C\left(\int_\Omega |\epsilon(u)|^2 \, dx + \int_D |u|^2 \, dx\right).$$

Boundary Korn Inequality

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Lemma (Dahlberg, Kenig, Verchota)

Let u be a solution of the Lamé system, $Lu = 0$ in Ω. Then

$$\int_{\partial\Omega} |\nabla u|^2 \, d\sigma \leq C \left(\int_{\partial\Omega} |\epsilon(u)|^2 \, d\sigma + \int_{\Omega} |u|^2 \, d\sigma \right) .$$

Two Poincaré inequalities

If u is in $H^1(\Omega)$, we have

$$\int_\Omega |u|^2 \, dx \le C \left(\int_\Omega |\nabla u|^2 \, dx + \int_D |u|^2 \, dx \right).$$

If u in $H^1(\partial\Omega)$ we have

$$\int_{\partial\Omega} |u|^2 \, d\sigma \le C \left(\int_{\partial\Omega} |\nabla_t u|^2 d\sigma + \int_D |u|^2 \, dx \right).$$

The main estimate

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Theorem

Let u be a solution of $Lu = 0$ and suppose that $(\nabla u)^* \in L^2(\partial\Omega)$. Let $a^{ij}_{\alpha\beta}$ satisfy the ellipticity condition (3) and symmetry condtion (4), then we have the estimate

$$\int_{\partial\Omega} |\nabla u|^2 \, d\sigma \leq C \left(\int_N \left| \frac{\partial u}{\partial \rho} \right|^2 d\sigma + \int_D |\nabla_t u|^2 + |u|^2 \, d\sigma \right). \tag{5}$$

A sketch of the proof

Using the change in sign (see (2) of $h \cdot \nu$ as we move from D to N, we have the lower bound

$$
\begin{aligned}
c \int_{\partial\Omega} |\epsilon(u)|^2 \, dx \quad \leq & \quad \int_N h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \, d\sigma \\
& - \int_D h_k \nu_k a^{ij}_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \, d\sigma
\end{aligned}
$$

The Rellich identity allows us to write the right-hand side of this inequality in terms of the data for the mixed problem. The boundary Korn inequality allows us to estimate the full gradient in terms of $\epsilon(u)$. The Korn and Poincaré inequalities help handle some lower order terms.

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Two families of operators

We will consider two families of coefficients,

$$a^{ij}_{\alpha\beta} = \mu\delta_{ij}\delta_{\alpha\beta} + (\lambda + \mu - r)\delta_{i\alpha}\delta_{j\beta} + r\delta_{i\beta}\delta_{j\alpha}, \quad 0 \le r \le \mu. \quad (6)$$

Note that this family of coefficients gives the Lamé system for each r. However, the boundary operator $\partial/\partial\rho$ will change with r.
We also consider the family

$$\mu\delta_{ij}\delta_{\alpha\beta} + r\delta_{i\alpha}\delta_{j\beta}, \qquad \lambda + \mu \ge r \ge 0.$$

Which connects the Laplacian to one form of the Lamé operator.

The regularity problem

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Suppose f_D is in $H^1(\partial\Omega)$. According to the work of Dahlberg, Kenig and Verchota, we may find a unique solution to the regularity problem for Lamé system. This was extended to general elliptic operators by W. Gao (1991).

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = f_D & \text{on } \partial\Omega \\ (\nabla u)^* \in L^2(d\sigma) \end{cases}$$

The Dirichlet to Neumann map

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Let L be an elliptic operator and $\partial/\partial\rho$ one of the Neumann-type boundary operators. If u is the solution of the regularity problem for L with Dirichlet data f, then we define

$$\Lambda f = \frac{\partial u}{\partial \rho}.$$

Given a solution of the regularity problem, solving the mixed problem is equivalent to showing

$$\Lambda : H^1_0(N) \to L^2(N)$$

is onto. The map Λ is one-to-one since energy methods give that the solution to the mixed problem is unique.

The main theorem

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Theorem (with I. Mitrea)

Let Ω be a creased domain as described above. Let L be the Lamé operator with $\mu > 0$ and $\lambda \geq 0$. If the Neumann data, f_N is in $L^2(N)$ and the Dirichlet data f_D is in the Sobolev space $H^1(D)$, then the mixed problem (with the traction boundary condition N) has a unique solution.

The last step in the proof.

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Let $\Lambda(r)$ denote the Dirichlet to Neumann map for the families of coefficients introduced above. According to our main estimate we have

$$\|f\|_{H^1_0(N)} \leq C\|\Lambda(r)f\|_{L^2(N)}$$

We know Λ is invertible for the Laplacian. Thus, by the method of continuity, Λ is invertible for the Lamé system with traction boundary condition.

The mixed problem or Zaremba's problem for the Laplacian

Joint work with L. Lanzani and L. Capogna. To appear in *Math. Ann.*
We consider the mixed problem for Laplace's equation.

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = f_D, & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N, & \text{on } N \\ (\nabla u)^* \in L^p(d\sigma) & \end{cases}$$

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The domains

In our results for Laplace's equation, the domain Ω satisfies

- $\Omega = \{x_2 > \phi(x_1)\}$, with $\phi : \mathbf{R} \to \mathbf{R}$
- $\|\phi'\|_\infty < 1$, this condition is probably too strong.
- $N = \{(t, \phi(t)) : t \geq 0\}$, $D = \{(t, \phi(t)) : t < 0\}$
- We will let $d\sigma_\epsilon = |x|^\epsilon d\sigma$ with $d\sigma$ denoting arc-length on the boundary of $\partial\Omega$.

A Rellich identity

Lemma (L. Escauriaza)

Suppose Ω is a graph Lipschitz domain and let $\alpha = c(\mathrm{Re}(x_1 + ix_2)^\epsilon, \mathrm{Im}(x_1 + ix_2)^\epsilon)$. If u is harmonic and $(\nabla u)^*$ lies in $L^2(|z|^\epsilon \, d\sigma)$, we have

$$\int_{\partial\Omega} |\nabla u|^2 \alpha \cdot \nu - 2\frac{\partial u}{\partial \alpha}\frac{\partial u}{\partial \nu} \, d\sigma = 0.$$

The proof uses that α is holomorphic.

Weighted estimates from the Rellich identity

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From the Rellich identity and a bit of trigonometry, we can prove.

Theorem

Let Ω, N and D be as above. Suppose that $\beta = \arctan(\|\phi'\|_\infty)$. Let u be harmonic in Ω and suppose $(\nabla u)^* \in L^2(d\sigma_\epsilon)$. If we have $2\beta/(\pi - 2\beta) < \epsilon < 1$, then

$$\int_{\partial\Omega} |\nabla u|^2 \, d\sigma_\epsilon \leq C \left(\int_N \left(\frac{\partial u}{\partial \nu}\right)^2 d\sigma_\epsilon + \int_D \left(\frac{du}{d\sigma}\right)^2 d\sigma_\epsilon \right)$$

The condition on ϵ and β comes when we try to find a vector field α so that $\nu \cdot \alpha > \delta |x|^\epsilon$ on N and $\nu \cdot \alpha < -\delta |x|^\epsilon$ on D.

The regularity problem with weights

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We will need to consider the regularity problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega \\ (\nabla u)^* \in L^2(d\sigma_\epsilon) \end{cases}$$

The following theorem on the regularity problem implies that it is sufficient to consider the case when $f_D = 0$.

Theorem (Shen, 2005)

Let $0 \le \epsilon < 1$ and Ω be the domain lying above the graph of a Lipschitz function. Suppose that $\nabla_t f$ is in $L^2(d\sigma_\epsilon)$. There exists exactly one solution of the regularity problem with $(\nabla u)^* \in L^2(d\sigma_\epsilon)$.

Solution of the weighted L^2 problem

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Theorem

Let $2\beta/(\pi - 2\beta) < \epsilon < 1$ and Ω, N and D be as above. We may find a solution of the mixed problem with

$$\|(\nabla u)^*\|_{L^2(d\sigma_\epsilon)} \leq C(\|f_N\|_{L^2(N,d\sigma_\epsilon)} + \|\frac{df}{d\sigma}\|_{L^2(D,d\sigma_\epsilon)}).$$

There is only one solution with $(\nabla u)^* \in L^2(d\sigma_\epsilon)$.

Hardy spaces

Let $\Delta_r(x_0) = \{x : x \in \partial\Omega, |x - x_0| < r\})$ for $r > 0$ and $x_0 \in \partial\Omega$. We say a is an atom for the Hardy space $H^1(d\sigma_{\epsilon'})$ if

- $\operatorname{supp} a \subset \Delta_r(x_0)$ for some $x_0 \in \partial\Omega$ and $r > 0$,
- $\int_{\partial\Omega} a\, d\sigma = 0$,
- $\|a\|_{L^\infty} \leq 1/\sigma_{\epsilon'}(\Delta_r(x_0))$

The Hardy space $H^1(d\sigma_{\epsilon'})$ is collection of functions of the form $\sum \lambda_j a_j$ with each a_j and atom and $\{\lambda_j\}$ in ℓ^1. The Hardy space on N, $H^1(N, d\sigma_{\epsilon'})$ consists of restrictions to N of functions in $H^1(d\sigma_{\epsilon'})$.

Existence in Hardy spaces.

Theorem

Suppose Ω, N and D are as above and that we may solve the $L^2(\sigma_\epsilon)$ mixed problem. Let u solve the mixed problem with zero Dirichlet data and with data $a|_N$ on N. There exists ϵ_0 so that for $|\epsilon'| < \epsilon_0$, we the estimate

$$\int_{\partial\Omega} (\nabla u)^* \, d\sigma_{\epsilon'} \leq C.$$

Ingredients of the proof

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- The L^2 theory gives us a solution. We need to establish the integrability.
- We construct a Green function for the mixed problem using the method of images.
- de Giorgi, Nash, Moser theory gives Hölder continuity for the Green function. This continuity and our assumption that a has mean value zero allows us to conclude that u decays at infinity.
- This decay implies the integrability of the $(\nabla u)^*$.
- From the estimate for atomic data it follows that we have a result for data with f_N and $df_D/d\sigma$ in an appropriate Hardy space.

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The result for $L^p(d\sigma)$

Theorem (with Lanzani and Capogna)

Let Ω, N and D be as above. There exists p_0 which depends only on the Lipschitz constant so that we may solve the mixed problem in $L^p(d\sigma)$ for $1 < p < p_0$.

The proof amounts to interpolating between the $L^2(d\sigma_\epsilon)$ result with $\epsilon > 0$ and the $H^1(d\sigma_{\epsilon'})$ result with $\epsilon' < 0$. If you have good aim, you will hit an unweighted L^p result. Stromberg and Torchinsky (1989) provide us with the necessary interpolation result.

Other results

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- A treatment a mixed problem for the stationary Stokes system in L^2 is given by Brown, I. Mitrea, M. Mitrea, M. Wright. (to appear, *Trans. Amer. Math. Soc.*)
- M. Venouziou and G. Verchota (2008) treat some instances of the mixed problem when the boundary between D and N is more complicated. For example, there work includes the case of a four-sided pyramid in three dimensions where we alternate between Dirichlet and Neumann conditions on the faces.

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Some questions

- What is true without the crease condition in three dimensions?
- What L^p-results can we obtain for the mixed problem for the Lamé system?
- Can we remove the restriction that the Lipschitz constant is at most 1 in the two dimensional result of Lanzani, Capogna and Brown?

Thanks

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- Thanks to the organizers for the invitation.
- Thanks to the AMS for showing the good sense in selecting Zhongwei to speak.
- In the unlikely event that you would like a copy of these slides, visit http://www.math.uky.edu/∼rbrown/conferences/