1 Lectures at ICMat/UAM, May 2011, Russell Brown

1.1 The inverse conductivity problem with less regular conductivities

We will consider the inverse conductivity problem or Calderón problem [11] in three dimensions (and higher). The fundamental paper is the work of Sylvester and Uhlmann [23] which gives global uniqueness in the inverse conductivity problem in three dimensions. In this problem we want to find a coefficient in an elliptic differential operator

The first step will be to recover the coefficient at the boundary. Boundary identifiability has been considered by a number of authors including Kohn and Vogelius [15], Sylvester and Uhlmann [24], Kang and Yun [14]. [1] My work gives identifiability for conductivities that have no smoothness [7]. With Salo, we applied this technique to identify the coefficient of a first order term [10].

After the discussion of boundary identifiability, we will discuss the method of Sylvester and Uhlmann paying close attention to the regularity hypotheses needed to carry out the argument. The results we discuss will include work of Brown [8], Panchenko, Päivärinta and Uhlmann [18] and Brown and Torres [9]. We mention, but will not discuss, a recent interesting paper of Horst [13] gives stability for conductivities with 3/2 derivatives.

These results require the conductivity to have 3/2 derivatives and no one believes that this is optimal. In two dimensions, Astala and Päivärinta [2] have given a method for recovering the conductivity that is only bounded and measurable. A construction due to Tolmasky [25], see also the work of Panchenko, Päivärinta and Uhlmann [18] allows us to construct exponentially growing solutions. We will give a version of this construction for the conductivity equation and indicate why this is not enough to allow recovery of the conductivity.

1.2 Scattering for a first order system

We consider a first order system in the plane and discuss the direct and inverse scattering transforms for this system. This material dates back to the 1980's and may be found in various forms in work of Fokas and Ablowitz [12], [3] and the work of L. Sung [20, 21, 22]. The inverse scattering problem is of interest because it serves to transform solutions of the focussing and de-focussing Davey-Stewartson II equations to a linear evolution.

We will construct the scattering transform and its inverse and develop their main properties as maps on Schwartz functions. This is a summary of earlier work. Our main goal is to describe two situations where the inverse scattering transform is known to be an isometry a) on a neighborhood of L^2 [6] and b) the space $H^{1,1}(\mathbf{R}^2) = \{q :$

$q, \nabla q, |\cdot| q \in L^2(\mathbf{R}^2)\}$ [19]. Case a) was studied by Brown and the second case is recent work of Peter Perry. I thank Peter for giving me access to a draft of his work. In the first case, we obtain results for the focusing and the defocussing equation. The second case covers only the defocussing equation. Examples due to Ozawa [17] show that a smallness condition is needed in the focussing case.

Both of these estimates rely on studying a family of multi-linear forms of the sort studied by Brascamp and Lieb. We give an estimate for these forms proved by Brown [6] with a second proof given in Nie and Brown [16]. We will not attempt to summarize the literature on Brascamp-Lieb forms, but mention the fundamental work of Brascamp and Lieb [5] and a recent paper of Bennett, Carbery, Christ and Tao [4].

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