The Green function for the mixed problem for the Stokes system

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Thanks

This talk reports joint work with Katharine Ott and Seick Kim.

Outline

The Stokes system

We let $\Omega \subset \mathbf{R}^2$ be a Lipschitz domain and assume we have a decomposition of the boundary $\partial\Omega = D \cup N$. We consider the linearized Stokes system in Ω with mixed boundary conditions.

$$\begin{cases} -\Delta u + \nabla p = f, & \text{in } \Omega \\ -\operatorname{div} u = g, & \text{in } \Omega \\ 2\epsilon(u)\nu - p\nu = f_N, & \text{on } N \\ u = f_D, & \text{on } D \end{cases}$$

Here, $u : \Omega \to \mathbf{R}^2$ represents the velocity of a fluid and p represents the pressure. We use $\epsilon(u) = (\nabla u + \nabla u^t)/2$ for the symmetric part of the gradient and ν represents the normal.

Formulation of the weak mixed problem for Stokes–L^2

We will use a weak formulation. We let $L^q_{1,D}(\Omega)$ denote the Sobolev space of functions which vanish on a subset $D \subset \partial\Omega$ and then we set $S_q = L^q_{1,D}(\Omega) \times L^q(\Omega)$. We consider a weak notion of solution to our boundary value problem in the special case that $f_D = 0$.

$$
\begin{aligned}
&a(u,\phi) - \int_\Omega p \operatorname{div} \phi \, dy \\
&\qquad = (f,\phi) + (f_N,\phi)_{\partial\Omega} - \int_\Omega \nabla g \cdot \phi \, dy, \qquad \phi \in L^{q'}_{1,D}(\Omega) \\
&-\operatorname{div} u = g \\
&(u,p) \in S_q
\end{aligned}
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The form $a(u,\phi)$ is defined by $a(u,\phi) = \int_\Omega \epsilon(u) \cdot \epsilon(\phi) \, dy$. Following classical arguments and especially work of Maz'ya and Rossmann we can show the existence of solutions to this boundary value problem for $q = 2$.

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Definition of the Green function

We give the following notion of Green function for the mixed problem for Stokes.

We say that $(G^{\alpha,\beta}(x,y))_{\alpha,\beta=1,2}, (\Pi^{\alpha}(x,y))_{\alpha=1,2}$ is a *Green function with pole at x* if whenever u is a solution of the weak mixed problem with data f and g in $C_c^\infty(\Omega)$ and $f_d = 0$ and $f_N = 0$, then we have that

$$u^\alpha(x) = \int_\Omega G^{\alpha\beta}(x,y) f^\beta(y) + \Pi^\alpha(y) g(y)\, dy.$$

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Lorentz spaces

A useful tool in our construction will be the Lorentz spaces, $L^{q,r}$, $(q, r) \in (1, \infty) \times [1, \infty]$.

- The Lorentz space $L^{q,q}$ is the usual Lebesgue space, L^q, for $1 < q < \infty$.
- The Lorentz spaces arise as real interpolation spaces of the Lebesgue spaces.
- For $1 \leq q < \infty$, we have that $|x|^{-n/q}$ lies in $L^{q,\infty}$.

To see why these spaces are useful, recall that the Green function for the Laplacian is $\frac{1}{\pi} \log |x|$. This suggests we should look for our Green function in a space of functions with derivative in the Lorentz space $L^{2,\infty}(\Omega)$. We will use the notation of $L_1^{q,r}(\Omega)$ for the Sobolev space of functions with gradient in $L^{q,r}(\Omega)$.

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Main theorem

Theorem

We assume that Ω, N and D satisfy the conditions,

a) *For $x \in D$ and $0 < r < r_0$, we have $\sigma(B_r(x)) \approx r$.*
b) *D and N have nonempty interior.*

We may find a Green function for the mixed problem which satisfies $\nabla . G(x,\cdot), \Pi(x,\cdot) \in L^{2,\infty}(\Omega)$ and

$$\begin{aligned} |G(x,y)| &\leq C(1+\log(d/|x-y|)) \\ |G(x,y)-G(x,z)| &\leq C\left(\frac{|y-z|}{|x-y|}\right)^{\gamma}, \qquad 2|y-z| < |x-y| \end{aligned}$$

Here, d denotes the diameter of Ω and the Hölder exponent $\gamma > 0$.

Solving the mixed problem in Lorentz spaces

We define a map $T : S_q \to S'_{q'}$ by $T(u,p) = (\lambda, \mu)$ by
$\lambda(\phi) = a(u,\phi) - \int p \operatorname{div} \phi \, dy$ and $\mu(h) = -\int_\Omega h \operatorname{div} \phi \, dy$.
The standard L^2-theory (see especially Mazya and Rossman [MR07]) for the Stokes operator tells us that $T : S_2 \to S'_2$ is invertible. A well-known perturbation argument shows that invertibility is preserved under small changes of a complex interpolation parameter (see Šneĭberg [Šne74] and Tabacco Vignati and Vignati [TVV88]). This gives solvabilty for q near 2.
Finally, real interpolation tells us that the operator T^{-1} is bounded as a map $T^{-1} : S'_{q',r'} \to S_{q,r}$.

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Interpolation

The argument relies on studying interpolation not only for the Lorentz Sobolev spaces, but also the subspace $L^{q,r}_{1,D}(\Omega)$. This has been considered before us by Auscher, Badr, Haller-Dintelmann, and Rehberg [ABHDR], Brewster and Mitrea[3] [BMMM], and Haller-Dintelmann, Jonsson, Knees, and Rehberg [HDJKR12].

The assumptions on the boundary and especially the Ahlfors regularity of arc-length restricted to *D* are needed in this argument.

Existence of Green function

We recall that functions with gradient in the Sobolev space of functions in $L^2_1(\Omega)$ are continuous and thus the Dirac delta measure is in the dual of this Lorentz Sobolev space.

More precisely we have that $(e_\alpha \delta_x, 0)$ lies in $S'_{2,1}$ and we put

$$(G^{\alpha\cdot}(x,\cdot), \Pi^\alpha(x,\cdot)) = T^{-1}(e_\alpha \delta_x, 0).$$

This argument is based on ideas of D. Mitrea and I. Mitrea [MM11].

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Reverse Hölder estimates

The solvability of the boundary value problem for Stokes in S_q for $q > 2$ gives us the following reverse Hölder inequality for solutions: We find a family of star-shaped Lipschitz domains $\Omega_\rho(x)$ which are star-shaped Lipschitz domains with scale ρ and Lipschitz constant M so that

$$\begin{aligned} &\|p\|_{L^q(\Omega_\rho(x)} + \|\nabla u\|_{L^q(\Omega_\rho(x)} \\ &\leq \quad C\|\eta f\|_{L^q_1(\Omega_{4\rho}(x))'} + \frac{1}{\rho}(\|p\|_{L^{\tilde{q}}(\Omega_{4\rho}(x))} + \|\nabla u\|_{L^{\tilde{q}}(\Omega_{4\rho}(x))}) \end{aligned}$$

Here $\frac{1}{\tilde{q}} = \frac{1}{q} + \frac{1}{2}$.

It is important that we have the solvability of the mixed problem in all of the domains $\Omega_\rho(x)$ for all small ρ. It is important that hypothesis a) will hold at all small scales. We only need to impose hypothesis b) at the scale of the domain. The argument involves studying ηu where η is a cutoff function and we have some freedom to to assign the boundary conditions on the part of the boundary where ηu vanishes.

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Hölder continuity of solutions

The Sobolev space $L^q_1(\Omega)$ embeds into the space of Hölder continuous functions $C^\gamma(\bar{\Omega})$ with $\gamma = 1 - 2/q$. Thus, the Sobolev space regularity result above and a Caccioppoli inequality imply the local regularity result for solutions

$$|u(z) - u(y)| \le C \left(\frac{|y-z|}{\rho} \right)^\gamma \left(\fint_{\Omega_{2\rho}(x)} |u - \bar{u}_{x,\rho}|^2 \, dy \right)^{1/2} .$$

Local regularity of the Green function

Using the local regularity estimates we obtain the following estimates for the Green function:

$$\begin{aligned} |G(x,y)| &\leq C(1+\log(d/|x-y|)) \\ |G(x,y)-G(y,z)| &\leq C\left(\frac{|y-z|}{|x-y|}\right)^{\gamma}, \qquad 2|y-z|<|x-y|. \end{aligned}$$

Here, d is the diameter of Ω. As a step towards the proof of the second estimate, we show that $G(x,\cdot)$ is in a variant of the space BMO.

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Here, d is the diameter of Ω. As a step towards the proof of the second estimate, we show that $G(x,\cdot)$ is in a variant of the space BMO.

Applications

Green function estimates have been an important tool in my work with Taylor, Sykes, Ott, Lanzani, and Capogna [SB01, LCB08, OB13, TOB13] and in independent work of I. Mitrea and M. Mitrea [MM07]. These authors are interested in establishing the existence of the solutions to the mixed problem and obtaining non-tangential estimates on the gradient.

This argument originates in work of Dahlberg and Kenig [DK87]. We expect that the Green function estimates we have established will help to carry out this argument for the Stokes operator in two dimensions.

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Problems we can't solve

1. Dimensions larger than 2. I understand that dimension 3 may have more applications.
2. There is one more row of the Green function which (formally can be represented has $T^{-1}(0, \delta_x)$) and is used to represent the pressure. What estimates can we prove for this function?

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