The mixed problem

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The mixed problem

We consider the mixed boundary value problem or Zaremba's problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \\ (\nabla u)^* \in L^p(\partial\Omega) & \end{cases}$$

where $(\nabla u)^*$ is the non-tangential maximal function and ν is the outer unit normal to the boundary.
Recall that if v is defined in Ω and $P \in \partial\Omega$, we may define the non-tangential maximal function by

$$v^*(P) = \sup\{|v(x)| : |x - P| < 2dist(x, \partial\Omega)\}.$$

Some history-Laplace's equation

- Solutions exist in the Sobolev space $L^{2,1}(\Omega)$ under quite general assumptions.
- The Dirichlet problem in a Lipschitz domain was treated by Dahlberg (1977).
- The Neumann problem and a regularity result for the Dirichlet problem for Laplace's equation were treated by Jerison and Kenig (1982).
- The study of the mixed problem in Lipschitz domains was a problem posed by Kenig in his CBMS lecture notes (1991).

Some more history

- There is a large literature on problems on polygonal and polyhedral domains that I do not attempt to summarize.
- G. Savaré (1997) showed that a solution of the mixed problem for elliptic equations could be found in the Besov space $B_{\infty}^{3/2,2}(\Omega)$ of smooth domains.
- M. Venouziou and G. Verchota (2008) treat the mixed problem in polyhedral domains when the boundary between D and N is more complicated. For example, their work includes the case of a four-sided pyramid in three dimensions where we alternate between Dirichlet and Neumann conditions on the faces.
- The work reported today is a continuation of work with Capogna, Lanzani, I. Mitrea, M. Mitrea, Sykes, and M. Wright on the mixed problem (1994,2001,2008,2009).

An example

Consider the sector, $\Omega_\alpha = \{re^{i\theta} : 0 < \theta < \alpha\}$, let $N = \{re^{i\alpha} : r > 0\}$, and $D = \{r : r > 0\}$. Then the function $u(re^{i\theta}) = r^{\pi/(2\alpha)} \sin(\theta\pi/(2\alpha))$ will be a solution to the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } N \\ u = 0 & \text{on } D \end{cases}$$

Considering the behavior near zero, we see that for each $p > 4/3$, we can find a Lipschitz domain where we cannot hope to solve the mixed problem in L^p.

A necessary condition for D.

If we can solve (MP), then we have constructed an extension operator from the Sobolev space $L^{p,1}(D)$ into $L^{p,1}(\partial\Omega)$. Thus, we must impose some conditions on D to solve the mixed problem or require that the Dirichlet data have an extension which lies in the Sobolev space $L^{p,1}(\partial\Omega)$.

Our domains

We assume that Ω is a bounded, connected, open set and that the boundary is given locally as the graph of a Lipschitz function.

We write $\partial\Omega = D \cup N$ with $D \cap N = \emptyset$, D a relatively open subset of $\partial\Omega$. We let $\Lambda = \bar{D} \cap N$ and suppose that for each $x \in \Lambda$, we may find $r > 0$ and a coordinate system on $\mathbf{R}^n$ and Lipschitz functions $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ and $\psi : \mathbf{R}^{n-2} \to \mathbf{R}$ so that

$$
\begin{aligned}
\Omega \cap B_r(x) &= \{(x', x_n) : x_n > \phi(x')\} \\
D \cap B_r(x) &= \{(x_1, x'', x_n) : x_1 > \psi(x''),\ x_n = \phi(x')\} \\
N \cap B_r(x) &= \{(x_1, x'', x_n) : x_1 \le \psi(x''),\ x_n = \phi(x')\}
\end{aligned}
$$

Main theorem of Ott and Brown

Theorem (Ott, Brown)

Let Ω, N, D satisfy the above conditions. Then there exist a $p_0 > 1$ so that for $1 < p < p_0$, the mixed problem with $f_N \in L^p(N)$ and $f_D \in L^{p,1}(D)$ has a unique solution which satisfies the estimate

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \le C(\|f_N\|_{L^p(N)} + \|f_D\|_{L^{p,1}(D)}).$$

Local Rellich estimates

Let $\Omega_r(x) = \Omega \cap B_r(x)$ and $\Delta_r(x) = \partial\Omega \cap B_r(x)$ where $B_r(x)$ denotes a ball with radius r and centered at x.
If u is a weak solution of the mixed problem and $\Delta_{2r}(x) \subset N$, then

$$\int_{\Delta_r(x)} |\nabla u|^2 \, d\sigma \leq C \int_{\Delta_{2r}(x)} |f_N|^2 \, d\sigma + \frac{C}{r} \int_{\Omega_{2r}(x)} |\nabla u|^2 \, dx.$$

If u is a weak solution of the mixed problem and $\Delta_{2r}(x) \subset D$, then

$$\int_{\Delta_r(x)} |\nabla u|^2 \, d\sigma \leq C \int_{\Delta_{2r}(x)} |\nabla_t f_D|^2 \, d\sigma + \frac{C}{r} \int_{\Omega_{2r}(x)} |\nabla u|^2 \, dx.$$

Weighted estimates

We let $\delta(x) = \mathrm{dist}(x, \Lambda)$. We make a Whitney decomposition of D and N, multiply the local estimates on each cube by the sidelength of the cube raised to the power $1-\epsilon$ and sum to obtain the estimate

$$\int_{\partial\Omega} |\nabla u|^2 \delta^{1-\epsilon} d\sigma \leq C \left(\int_N \delta^{1-\epsilon} |f_N|^2 \, d\sigma + \int_D \delta^{1-\epsilon} |\nabla_t f_D|^2 \, d\sigma + \int_\Omega \delta^{-\epsilon} |\nabla u|^2 \, dy \right).$$

Reverse Holder inequalities

We may use the solution of the regularity problem to reduce to the case when the Dirichlet data f_D is zero.
Adapting the Caccioppoli inequality to allow for nonzero Neumann data, we may use the reverse Hölder technique of Gehring and Giaquinta to find a $q_0 > 2$ so that for t in $[2, q_0)$, we have

$$\left(\frac{1}{r^n}\int_{\Omega_r(x)} |\nabla u|^t \, dx\right)^{1/t} \leq C\left(\frac{1}{r^n}\int_{\Omega_{2r}(x)} |\nabla u|^2 \, dx\right)^{1/2} + \left(\frac{1}{r^{n-1}}\int_{\Delta_{2r}(x)\cap N} f_N^{\frac{t(n-1)}{n}} \, d\sigma\right)^{\frac{n}{t(n-1)}}.$$

Green function for the mixed problem

Using our assumption on the boundary between D and N, we can construct a Green function for the mixed problem by the method of images.
As a consequence of the de Giorgi, Moser, Nash regularity theory this Green function is Hölder continuous and has the same singularity as the fundamental solution for the Laplacian.

Hardy space estimate

We say that a is an atom for $\partial\Omega$ if $\operatorname{supp} a \subset \Delta_r(x)$ for some $x \in \partial\Omega$ and $r > 0$, $\int_{\partial\Omega} a\, d\sigma = 0$ and $\|a\|_\infty \leq \sigma(\Delta_r(x))^{-1}$.
We say that a is an *atom* for N if a is the restriction to N of an atom.
Using the weighted boundary estimate and the reverse Hölder estimate we are able to obtain an estimate for solutions of the mixed problem when the Neumann data is an atom, and thus for Neumann data taken from a Hardy space.
This argument follows, more or less, the argument of Dahlberg and Kenig (1987) for the study of the Neumann problem in Hardy spaces.

L^p estimates

The above estimates and a modification of the techniques of Shen (2007) (and Caffarelli and Peral and . . .) gives L^p estimates for p near 1.

Hypotheses on Λ

We assume that
a) Λ is regular in the sense that for each x in Λ and $0 < r < \mathrm{diam}(\Omega)$.

$$\mathcal{H}^{n-2}(\Lambda \cap \Delta_r(x)) \approx r^{n-2}$$

b) For each x in D and $0 < r < r_0$ we may find x_D with $|x - x_D| \approx r$ and $\delta(x_D) \approx r$.
c) For each x in N and $0 < r < r_0$, we may find x_n with $|x - x_N| \approx r$ and $\delta(x) \approx r$.
Condition b) gives estimates for the Green function–this is based on work of Stampacchia (1964).

Our next result

Theorem (Taylor, Ott, Brown)

Suppose that D, Λ and N satisfy the hypotheses of the previous slide. Assume that f_N lies in $L^p(N)$ and f_D is the restriction to D of a function in $L^{p,1}(\partial\Omega)$. There is a $p_0 > 1$ so that for $1 < p < p_0$, the mixed problem has a unique solution which satisfies the estimate

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C(\|f_N\|_{L^p(N)} + \|f_D\|_{L^{p,1}(D)}).$$

The end

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