The mixed problem in two-dimensional Lipschitz domains.

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The mixed problem.

We consider domains $\Omega = \{(x_1, x_2) : x_2 > \phi(x_1)\}$ where $\phi$ satisfies $\|\phi'\|_\infty < \infty$ and, for convenience, $\phi(0) = 0$. We write $\partial \Omega = D \cup N$ with $D = \partial \Omega \cap \{x_1 > 0\}$ and $N = \partial \Omega \cap \{x_1 \leq 0\}$. We consider the following classical boundary value problem,

$$
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u = f_D & \text{on } D \\
\frac{\partial u}{\partial \nu} = f_N & \text{on } N \\
(\nabla u)^* \in L^p(\omega \, d\sigma)
\end{cases}
$$

We will generally require data $f_N$ from $L^p(N, \omega \, d\sigma)$ and ask that $df_D/ds$ lie in $L^p(D, \omega \, d\sigma)$. The weights $\omega$ will be of the form $|x|^\epsilon$.

If $p = 1$, we must replace the $L^p$-spaces by Hardy spaces.
The non-tangential maximal function

Our estimate for $\nabla u$ uses the non-tangential maximal function, $(\nabla u)^*$. For $x \in \Omega$, we define $\Gamma(x) = \{y : A|y_1 - x_1| < (y_2 - x_2)\}$ for some $A$, $\|\phi\|_\infty < A < \infty$. Then for a function $u$ on $\Omega$, we define the non-tangential maximal function by

$$u^*(x) = \sup_{y \in \Gamma(x)} |u(y)|.$$  

This function allows us to use the dominated convergence theorem to show that the values of $u$ on surfaces parallel to the boundary converge as we approach the boundary.
An example

If we let \( u(r, \theta) = r^{1/2} \cos(\theta/2) \), then

\[
\begin{cases}
\Delta u = 0 & \text{in } \{(x_1, x_2) : x_2 > 0\} \\
u(x_1, 0) = 0, & x_1 < 0 \\
\frac{\partial u}{\partial \nu}(x_1, 0) = 0, & x_1 > 0
\end{cases}
\]

but we do not have \( \nabla u(x_1, 0) \in L^2_{\text{loc}}(\mathbb{R}) \).

Thus, we cannot solve the mixed problem in \( L^2 \) with respect to arc-length for smooth domains. (We focus on the local behavior of \( u \) and ignore the behavior at infinity.)
Some results

1. Azzam and Kreyszig (1982) found solutions in $C^{2+\alpha}(\bar{\Omega})$ provided $\Omega$ is piecewise smooth and $N$ and $D$ meet at sufficiently small angles.

2. Savaré (1997) shows that in a smooth domain, the solution lies in the Besov space $B^{3/2,2}_{\infty}(\Omega)$. (The estimate $\nabla u \in L^2(\partial\Omega)$ is equivalent to $u \in B^{3/2,2}_2$.)

3. RB (1994) studied Lipschitz domains where $N$ and $D$ meet at an angle strictly less than $\pi$ and established existence of solutions for $L^2(\partial\Omega)$ with respect to surface measure.

The main theorem

**Theorem 1** Let \( \| \phi' \|_{\infty} < 1 \). There exists a value \( p_0 = p(\| \phi' \|_{\infty}) > 1 \) so that if \( 1 < p < p_0 \), then the mixed problem with data \( f_N \in L^p(N) \) and \( df_D/ds \in L^p(D) \) has a unique solution which satisfies

\[
(\nabla u)^* \in L^p(\partial \Omega, d\sigma).
\]
Weighted $L^2$ estimates from the Rellich-(Payne-Weinberger-Pohozaev-....) identity

As we observed above, the mixed problem is not solvable in $L^2$ for a half-space. However, we can study the problem in weighted $L^2$ spaces where the weight is of the form $|x|^\epsilon$. The basic estimate will be obtained from the following extension of the Rellich identity which we learned from Luis Escauriaza.

**Lemma 1 (Rellich Identity)** Let $\epsilon > -1$ and $\alpha(z) = az^\epsilon \equiv (\text{Re}(az^\epsilon), \text{Im}(az^\epsilon))$ for some $a \in \mathbb{C}$. Here, $z = x_1 + ix_2 \equiv (x_1, x_2)$. If $u$ is harmonic in $\Omega$ and $(\nabla u)^* \in L^2(\sigma_\epsilon)$, then we have

$$
\int_{\partial \Omega} |\nabla u|^{2\alpha} \cdot \nu - 2\alpha \cdot \nabla u \frac{\partial u}{\partial \nu} \, d\sigma = 0.
$$

To establish this lemma, we observe that $\text{div}(|\nabla u|^{2\alpha} - 2\nabla u \cdot \alpha \nabla u) = 0$ and use the divergence theorem.
Use of the Rellich identity

To make use of this Lemma, we will need to find a vector field \( \alpha = (\text{Re} az^\epsilon, \text{Im} az^\epsilon) \) so that

\[
\begin{align*}
\alpha \cdot \nu & \geq c|z|^\epsilon, \quad \text{on } N \\
\alpha \cdot \nu & \leq -c|z|^\epsilon, \quad \text{on } D.
\end{align*}
\]

Then if we rearrange the terms, we will obtain

\[
\int_D \left| \frac{du}{ds} \right|^2 d\sigma_\epsilon + \int_N \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma_\epsilon \approx \int_N \left| \frac{du}{ds} \right|^2 d\sigma_\epsilon + \int_D \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma_\epsilon
\]

This is the main estimate we need for existence.
Trigonometry

Lemma 2 Let $\beta = \arctan \|\phi'\|_{\infty} > 0$. Assume $\beta < \pi/4$. Then, for $2\beta/(\pi - 2\beta) < \epsilon < 1$ there exist $\beta_0 = \beta_0(\epsilon, M)$, $\beta < \beta_0 < (\pi - 2\beta)/2$, and a complex number $a = e^{i\lambda}$ such that the vector field $\alpha(x) = (\text{Re}(az^\epsilon), \text{Im}(az^\epsilon))$ satisfies

$$-|x|^{\epsilon} \leq \alpha(x) \cdot \nu(x) < -|x|^{\epsilon} \sin(\beta_0 - \beta), \quad x \in N; \quad (1)$$

$$|x|^{\epsilon} \geq \alpha(x) \cdot \nu(x) > |x|^{\epsilon} \sin(\beta_0 - \beta), \quad x \in D. \quad (2)$$
The restriction $M < 1$

We continue to use $\alpha = (\text{Re} \, az^\epsilon, \text{Im} \, az^\epsilon)$.

Suppose $\Omega$ is a graph domain with Lipschitz constant of 1.

If $\alpha \cdot \nu > 0$ on $N$, we must have $\arg \alpha(x) \in (-3\pi/4, -\pi/4)$ for $\arg(x) \in (-\pi/4, \pi/4)$. Thus $\epsilon < 1$.

If $\alpha \cdot \nu < 0$ on $D$, we must have $\arg \alpha(x) > \pi/4$ when $\arg x = 3\pi/4$. Thus $\epsilon > 1$. 
Hardy spaces

We recall the atomic definition of weighted Hardy spaces, $H^1(\sigma_{\epsilon'})$.

We say $a$ is an atom for $H^1(\sigma_{\epsilon'})$ if (i) $a$ is supported in an interval on $\partial \Omega$, $I$, (ii) $\int_I a \, d\sigma = 0$ and (iii) $\|a\|_{\infty} \leq \sigma_{\epsilon'}(I)^{-1}$.

An element of the Hardy space is given by $\sum \lambda_j a_j$ where the coefficients satisfy $\sum |\lambda_j| < \infty$. If $A$ is a subset of $\partial \Omega$, we define the Hardy space $H^1(A, \sigma_{\epsilon'})$ as the restrictions of $A$ of elements of the Hardy space on $\partial \Omega$. 
The behavior of the Green’s function and a related Hardy space estimate

We can define a Green’s function for the mixed problem and it is Hölder continuous as follows:

\[ |M(x, y_1) - M(x, y_2)| \leq C \left( \frac{|y_1 - y_2|}{|x - y_1|} \right) ^\delta , \quad 2|y_1 - y_2| < |x - y_1|. \]

This estimate can be used to prove that if \( u \) is a solution of the mixed problem with Neumann data an atom and zero Dirichlet data then \( |\nabla u| \) decays like \( |x|^{-1-\delta} \) at infinity (at least in an average sense). This is the key estimate that we need to show that

\[(\nabla u)^* \in L^1(\sigma_{\epsilon'}).\]

We show that we can solve the mixed problem for data \( f_N \in H^1(N, \sigma_{\epsilon'}) \) and \( df_D/ds \in H^1(D, \sigma_{\epsilon'}) \) for \(-\epsilon_0 < \epsilon' \leq 0\).
Interpolation, an $L^p$-result

Stromberg and Torchinsky (1989) have established a theorem on interpolation for Hardy spaces which allows change of measure. Thus interpolation between $L^2(\sigma_\epsilon)$ with $\epsilon > 0$ and $L^2(\sigma_{\epsilon'})$ with $\epsilon' < 0$, we conclude that we may solve the mixed problem with data in (unweighted) $L^p(\sigma)$ for $1 < p < p_0$. 
Two questions.

1. Is the restriction to \( M < 1 \) essential?

2. Can we establish existence for \( L^p \), with \( p \) near 1 in Lipschitz domains in higher dimensions?

Shen has a method for obtaining weighted estimates that does not rely on complex analysis.